

The spectral theory of generalized Laplacians associated to integrable metrics on compact Riemann surfaces

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Abstract

We extend the spectral theory of generalized Laplacians to integrable metrics on compact Riemann surfaces. As a consequence, we attach in a direct way, a holomorphic analytic torsion to any integrable metrics. We also provide a different approach to define the analytic torsion. We prove that both approaches agree.

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1 Introduction

The main goal of this paper is the extension, in the context of the Arakelov geometry, of the notion of the holomorphic analytic torsion to a large class of singular metrics, with applications to the special case of canonical metrics on \mathbb{P}^1 , viewed as toric variety. As an application, we provide an explanation for the results and the computations in [10] and [11].

Let X be a compact Riemann surface, and $(\mathcal{O}, h_{\mathcal{O}})$ the trivial line bundle equipped with a constant metric over X . In this article, we prove that the classical spectral theory of Laplacians can be extended to the class of integrable metrics on X . Namely, for any integrable metric $h_{X,\infty}$ on X , we construct a Laplacian attached to $((X, h_{X,\infty}); (\mathcal{O}, h_{\mathcal{O}}))$, and we denote it by $\Delta_{X,\infty}$. We show that $\Delta_{X,\infty}$ has a infinite discrete and positive spectrum, and we prove that the associated Zeta function $\zeta_{\Delta_{X,\infty}}$, has the

same properties as in the classical setting. In particular, we establish that $\zeta_{\Delta_{X,\infty}}$ admits a holomorphic continuation at $s = 0$. We set $T((X, \omega_{X,\infty}), (\mathcal{O}, h_{\mathcal{O}})) := \zeta'_{\Delta_{X,\infty}}(0)$, and we call it the holomorphic analytic torsion associated to $((X, \omega_{X,\infty}), (\mathcal{O}, h_{\mathcal{O}}))$. Moreover, given $(h_{X,p})_{p \in \mathbb{N}}$ a sequence of smooth hermitian metrics on X , converging uniformly in suitable way to $h_{X,\infty}$, we prove that the real sequence $(T((X, \omega_{X,p}), (\mathcal{O}, h_{\mathcal{O}})))_{p \in \mathbb{N}}$ converges to $T((X, \omega_{X,\infty}), (\mathcal{O}, h_{\mathcal{O}}))$.

Let us recall the construction of the holomorphic analytic torsion. For sake of simplicity, we restrict ourselves to compact riemannian surfaces. Let X be a compact riemannian surface equipped with a smooth metric h_X , we denote by ω_X the corresponding Kähler form, and let $(\mathcal{O}, h_{\mathcal{O}})$ be a trivial line bundle on X , endowed with a constant metric. We can equip $A^{(0,0)}(X)$, the space of smooth functions on X , with a hermitian product § (2), and then we consider Δ the Laplacian acting on $A^{(0,0)}(X)$ § (2). It is known that Δ admits a discrete, positive and infinite spectrum, see for instance [9, § 6]. Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the nonzero eigenvalues of Δ , counted with their multiplicities and order in increasing order. Using the spectral theory of heat kernels, see [2], one proves that for any $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$, the sum:

$$\zeta_{\Delta}(s) = \sum_{n \geq 1} \frac{1}{\lambda_n^s},$$

converges absolutely and admits a meromorphic continuation to the whole complex space. Moreover, this continuation is holomorphic at $s = 0$. According to Ray and Singer [17], we define the holomorphic analytic torsion associated to $((X, h_X), (\mathcal{O}, h_{\mathcal{O}}))$ as follows:

$$T((X, h_X), (\mathcal{O}, h_{\mathcal{O}})) = \zeta'_{\Delta}(0)^1.$$

As pointed out in [2], the smoothness of the metrics is a necessary condition in order to defined the holomorphic analytic torsion. In this article, we extend the latter theory to compact Riemann surfaces equipped with integrable metrics.

Let us review the contents of the article. Let X be a compact Riemann surface, and $(\mathcal{O}, h_{\mathcal{O}})$ the trivial line bundle equipped with a constant metric on X . Let $h_{X,\infty}$ be an integrable metric on X , see definition (4.1), we will construct a spectral theory attached to $h_{X,\infty}$ and $h_{\mathcal{O}}$, and which extend the classical theory. In section (2), we recall the definition and the construction of the Laplacian associated to smooth metrics on compact Riemann surfaces. We extend in section (3), this notion to any integrable metric $h_{X,\infty}$ on X , namely we construct a singular Laplacian denoted by $\Delta_{X,\infty}$ defined on $A^{(0,0)}(X)$. Our first theorem (theorem (3.3)) is stated as follows:

Theorem 1.1. *There exists $(h_{X,u})_{u \geq 1}$ a sequence of smooth hermitian metrics on X , converging uniformly to $h_{X,\infty}$ such that:*

1.

$$\lim_{u \rightarrow \infty} \|\Delta_{X,u} \xi\|_{L^2, u}^2 = \|\Delta_{X,\infty} \xi\|_{L^2, \infty}^2 < \infty,$$

2. $\Delta_{X,\infty}$ is a linear operator from $A^{(0,0)}(X)$ to $\mathcal{H}_0(X)$ (where $\mathcal{H}_0(X)$ is the completion of $A^{(0,0)}(X)$ with respect to the L^2 -norm).

3.

$$(\Delta_{X,\infty} \xi, \xi')_{L^2, \infty} = (\xi, \Delta_{X,\infty} \xi')_{L^2, \infty},$$

¹Notice that in [17], by definition $T((X, h_X), (\mathcal{O}, h_{\mathcal{O}}))$ is equal to $\zeta'_{\Delta_1}(0)$, where Δ_1 is the Laplacian associated to $((X, h_X); (\mathcal{O}, h_{\mathcal{O}}))$ and acting on $A^{(0,1)}(X)$. But from [19, (6) p.131], we have $\zeta'_{\Delta}(0) = \zeta'_{\Delta_1}(0)$.

4.

$$(\Delta_{X,\infty}\xi, \xi)_{L^2, \infty} \geq 0,$$

for any $\xi, \xi' \in A^{0,0}(X)$.

Since $h_{X,u}$ is smooth, a classical fact, see [9, p.94], asserts that $(I + \Delta_{X,u})^{-1}$ is a compact operator on $\mathcal{H}_0(X)$, for any $u \geq 1$. It is natural to ask if $I + \Delta_{X,\infty}$ is invertible in a suitable space. The answer to this question is given in the following theorem (theorem (3.8)):

Theorem 1.2. *The operator $\Delta_{X,\infty}$ admits a maximal selfadjoint extension to a space, denoted by $\mathcal{H}_2(X)$. We denote this extension also by $\Delta_{X,\infty}$. The operator $I + \Delta_{X,\infty}$ is invertible, more precisely, we have:*

$$(I + \Delta_{X,\infty})(I + \Delta_{X,\infty})^{-1} = I,$$

on $\mathcal{H}_0(X)$, where I is the operator identity of $\mathcal{H}_0(X)$.

$$(I + \Delta_{X,\infty})^{-1}(I + \Delta_{X,\infty}) = I,$$

on $\mathcal{H}_2(X)$, where I is the operator identity of $\mathcal{H}_2(X)$.

The proof of this theorem will be split into two steps, but before we need to establish two technical results (3.4) and (3.5). In the first step, we prove the following theorem (theorem (3.6)):

Theorem 1.3. *The sequence $((I + \Delta_{X,u})^{-1})_{u \geq 1}$ converges to a compact operator denoted by $(I + \Delta_{X,\infty})^{-1} : \mathcal{H}_0(X) \rightarrow \mathcal{H}_0(X)$, with respect to $L^2_{X,\infty}$ -norm.*

We extend the operator $\Delta_{X,\infty}$ in the second step, this is the goal of the subsection (3.2), where we review the notion of selfadjoint extension of Laplacians in the classical setting. Using theorem (3.6), we establish that $\Delta_{X,\infty}$ admits a maximal positive selfadjoint extension.

Next, we prove that $\Delta_{X,\infty}$ has a infinite discrete and positive spectrum, and by an operator theory argument it admits a heat kernel $e^{-t\Delta_{X,\infty}}$ for any $t > 0$. The following theorem (see (3.12)) shows that $e^{-t\Delta_{X,\infty}}$ is a limit of a sequence $(e^{-t\Delta_{X,u}})_u$ of heat kernels attached to $(h_{X,u})_{u \geq 1}$, a sequence of smooth metrics on X .

Theorem 1.4. *For any $t > 0$, we have:*

$$(e^{-t\Delta_{X,u}})_u \xrightarrow{u \rightarrow +\infty} e^{-t\Delta_{X,\infty}},$$

In particular, $e^{-t\Delta_{X,\infty}}$ is a compact operator from $\mathcal{H}_0(X)$ to $\mathcal{H}_0(X)$.

In order to prove this theorem, we show first that the sequence $(e^{-t\Delta_{X,u}})_u$ converges to a limit, for any $t > 0$. Then, we conclude using the uniqueness of the heat kernel.

In subsection (3.3), we study the spectral properties of $\Delta_{X,\infty}$. We introduce $\theta_{X,\infty}$, the associated Theta function and we prove, as in the classical theory, that $\theta_{X,\infty}(t)$ is finite for any $t > 0$, that is the theorem (3.16). The theorem (3.17) is the core of the article: If we denote by $(\lambda_{\infty,k})_{k \in \mathbb{N}}$ the sequence of the eigenvalues of $\Delta_{X,\infty}$ counted with their multiplicities, and ordered in increasing order, we have:

Theorem 1.5. *We have, for any $t > 0$ fixed:*

$$(\theta_{X,u}(t))_{u \geq 1} \xrightarrow{u \rightarrow \infty} \theta_{X,\infty}(t),$$

and

$$\zeta_{X,\infty}(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \theta_{X,\infty}(t) dt = \sum_{k=1}^\infty \frac{1}{\lambda_{\infty,k}^s},$$

is finite for any $s \in \mathbb{C}$, such that $\operatorname{Re}(s) > 1$. This function of s admits a meromorphic continuation to the whole complex plane with a pole at $s = 1$, and it is holomorphic at $s = 0$. We have,

$$\zeta'_{X,\infty}(0) = \int_1^\infty \frac{\theta_{X,\infty}(t)}{t} dt + \gamma b_{\infty,-1} - b_{\infty,0} + \int_0^1 \frac{\rho_{X,\infty}(t)}{t} dt = \lim_{u \rightarrow \infty} (\zeta'_{X,u}(0))_{u \geq 1},$$

where $b_{\infty,-1}, b_{\infty,0}$ are real numbers and $\rho_{X,\infty}$ is a real function such that $\theta_{X,\infty}(t) = \frac{b_{\infty,-1}}{t} + b_{\infty,0} + \rho_{X,\infty}(t)$, and $\rho_{X,\infty}(t) = O(t)$ for $t > 0$ sufficiently small.

The proof of this theorem relies on some technical lemmas, and on a critical result which gives a uniform lower bound for the first nonzero eigenvalue of $\Delta_{X,u}$ for any $u \geq 1$.

We provide in theorem (3.21) a new approach to extend the Quillen metrics to integrable metrics on compact riemannian surface; we prove the following result:

Theorem 1.6. *We keep the same assumptions. For any $p \in \mathbb{N}$, let $h_{Q,((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))}$ be the Quillen metric associated to $((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))$. We have, the sequence $\left(h_{Q,((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))}\right)_{p \in \mathbb{N}}$ converges to a limit, which does not depend on the choice of $(h_{X,p})_{p \in \mathbb{N}}$. We denote this limit by $h_{Q,((X,\omega_{X,\infty});(\mathcal{O},h_{\mathcal{O}}))}$.*

We finish this section, by comparing both methods. This is done in theorem (3.22), it is stated as follows :

Theorem 1.7. *We have,*

$$h_{Q,((X,\omega_{X,\infty});(\mathcal{O},h_{\mathcal{O}}))} = h_{L^2,((X,\omega_{X,\infty});(\mathcal{O},h_{\mathcal{O}}))} \exp(\zeta'_{X,\infty}(0)).$$

We recall and review in section (4) some classical notions used through this article. In section (5), we prove some technical results.

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2 The Laplacian on compact Riemann Surface

Let us recall the construction of the generalized Laplacian Δ acting on $A^{(0,0)}(X)$. We will emphasize that this construction does not require the smoothness of h_X , we can assume that h_X is only continuous.

Let h_X be a continuous hermitian metric on TX , and $h_{\mathcal{O}}$ a constant metric on \mathcal{O} . We denote by ω_X the normalized Kähler form associated to h_X , given on any local chart of X as follows:

$$\omega_X = \frac{i}{2\pi} h_X \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) dz \wedge d\bar{z}.$$

This metric induces a metric on the space of differential forms of type $(0,1)$. Tensoring by $h_{\mathcal{O}}$, the metric of \mathcal{O} , we obtain a pointwise inner product at any $x \in X$: $(s(x), t(x))$ for two section of

$A^{0,q}(X) = A^{0,q}(X) \otimes_{\mathcal{C}^\infty(X)} A^0(X)$, and $q = 0$ or 1 . The L^2 inner product of two sections $s, t \in A^{0,q}(X)$ is given by the formula:

$$(s, t)_{L^2} = \int_X (s(x), t(x)) \omega_X.$$

The Cauchy-Riemann operator $\bar{\partial}_{\mathcal{O}}$ acts on the forms of type $(0, q)$ with values in \mathcal{O} . We have the Dolbeault complex:

$$0 \longrightarrow A^{0,0}(X) \xrightarrow{\bar{\partial}_{\mathcal{O}}} A^{0,1}(X) \longrightarrow 0$$

Its cohomology is known to be the sheaf cohomology of X with coefficients in \mathcal{O} , cf. for example [9].

The operator $\bar{\partial}_{\mathcal{O}}$ admit a formal adjoint for the inner product L^2 ; namely an application

$$\bar{\partial}_{\mathcal{O}}^* : A^{0,1}(X) \longrightarrow A^{0,0}(X)$$

which verify

$$(s, \bar{\partial}_{\mathcal{O}}^* t)_{L^2} = (\bar{\partial}_{\mathcal{O}} s, t)_{L^2}.$$

for any $s \in A^{0,q}(X)$ et $t \in A^{0,q+1}(X)$. It follows from the definition that the operator $\bar{\partial}_{\mathcal{O}}^*$ is given by the formula:

$$\bar{\partial}_{\mathcal{O}}^* = - *_0^{-1} \bar{\partial}_{K_X \otimes \mathcal{O}^*} *_1,$$

see for instance [21, §.5], where $*_0$ and $*_1$ are the following applications:

$$*_0 : A^{0,0}(X) \longrightarrow A^{1,1}(X, \mathcal{O}^*),$$

and,

$$*_1 : A^{0,1}(X) \longrightarrow A^{1,0}(X, \mathcal{O}^*).$$

They are the unique applications which satisfy the following:

$$f(x) \wedge *_0(g(x)) = f(x) \overline{g(x)} \omega_x,$$

and

$$(f d\bar{z}) \wedge *_1(g d\bar{z}) = (f d\bar{z}(x), g d\bar{z}(x))_x \omega_X(x),$$

for any $x \in X$ such that $f, g \in A^{0,0}(X)$. Notice that in order to define $*_0$ and $*_1$, we do not need that h_X to be smooth. We can show easily that these morphisms can be written respectively on a local chart, as follows:

$$*_0(g) = \bar{g} \omega_X.$$

and,

$$*_1(g d\bar{z}) = -\bar{g} dz. \tag{1}$$

We denote by $\Delta_{\mathcal{O}}^0$, or by $\Delta_{\overline{\mathcal{O}}}$, or simply by Δ the operator $\bar{\partial}_{\mathcal{O}}^* \bar{\partial}_{\mathcal{O}}$ on $A^{0,0}(X)$. Following [19, Definition 8.1, p.101], we call it the generalized Laplacian associated to h_X and $h_{\mathcal{O}}$.

Remark 2.1. Even h_X is not smooth, the operator $\Delta_{\overline{\mathcal{O}}} = \bar{\partial}_{\mathcal{O}}^* \bar{\partial}_{\mathcal{O}}$ is well defined.

Lemma 2.2. *Let (X, h_X) be a compact Riemann surface such that h_X is continuous, and $(\mathcal{O}, h_{\mathcal{O}})$ the trivial line bundle equipped with a constant metric such that $h_{\mathcal{O}}$ is smooth. The Laplacian Δ associated to h_X and $h_{\mathcal{O}}$, is given locally as follows:*

$$\Delta(f) = -h_X \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right)^{-1} \frac{\partial^2 f}{\partial z \partial \bar{z}}, \quad (2)$$

for any $f \in A^{0,0}(X)$, where $\{\frac{\partial}{\partial z}\}$ is a local holomorphic basis of TX .

Proof. See for instance [14, definition 2.3.3]. □

3 The variation of metric on TX and the operator $\Delta_{X,\infty}$

In this section, we construct a singular Laplacian, $\Delta_{X,\infty}$ attached to an integrable metric on X , and we study its spectral properties.

3.1 The singular Laplacian $\Delta_{X,\infty}$

Let X be a compact riemannian surface, and we equip \mathcal{O} the trivial line bundle on X with constant metric $h_{\mathcal{O}}$ such that $h_{\mathcal{O}}(1, 1) = 1$. We endow X with an integrable metric $h_{X,\infty}$. By definition, there exist $h_{1,\infty}$ and $h_{2,\infty}$ two admissible metrics (see (4.3)) such that $h_{X,\infty} = h_{1,\infty} \otimes h_{2,\infty}^{-1}$. Let $(h_{1,n})_{n \in \mathbb{N}}$ and $(h_{2,n})_{n \in \mathbb{N}}$ be two sequences of smooth semipositive metrics which converge uniformly to $h_{1,\infty}$ and $h_{2,\infty}$, respectively. Let $h_{X,n} := h_{n,1} \otimes h_{2,n}^{-1}$ for any $n \in \mathbb{N}$, and we consider the family $(h_{X,u})_{u \geq 1}$ attached to this sequence as in (5.1). Recall that $h_{X,u}$ is a smooth hermitian metric on TX . We denote by $\omega_{X,u}$ the normalized volume form attached and by $\Delta_{X,u}$ the Laplacian attached to $h_{X,u}$ and $h_{\mathcal{O}}$ for any $u \in]1, \infty[$.

For all $u \in]1, \infty]$, we denote by $L_{X,u}^2$ (resp. $(\cdot, \cdot)_{L^2,u}$) the hermitian norm (resp. the hermitian product) induced by $h_{X,u}$ and $h_{\mathcal{O}}$ on $A^{(0,0)}(X)$, as in the previous section. We denote by $\mathcal{H}_0(X, u)$ the completion of $A^{(0,0)}(X)$ with respect to $L_{X,u}^2$ -norm.

Lemma 3.1. *The family $(L_{X,u}^2\text{-norms})_{u \geq 1}$ forms a sequence of uniformly equivalent norms on $A^{(0,0)}(X)$. In particular, $\mathcal{H}_0(X, u)$ does not depend on u , we will denote it by $\mathcal{H}_0(X)$.*

Proof. It suffices to notice that $(h_{X,u})_{u \geq 1}$ forms a bounded sequence, and we conclude using the compactness of X . □

Definition 3.2. For all $\xi \in A^{0,0}(X)$, we set:

$$\Delta_{X,\infty} \xi := -h_{X,\infty} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right)^{-1} \frac{\partial}{\partial z} \left(\frac{\partial \xi}{\partial \bar{z}} \right).$$

where $\{\frac{\partial}{\partial z}\}$ is a local holomorphic basis of TX . We say, $\Delta_{X,\infty}$ is the Laplacian attached to $h_{X,\infty}$ and $h_{\mathcal{O}}$.

In the following theorem, we prove that $\Delta_{X,\infty}$ is a linear operator defined over $A^{(0,0)}(X)$ with values belong to $\mathcal{H}_0(X)$.

Theorem 3.3. *Keeping the same hypothesis as before, we have:*

1.

$$\lim_{u \rightarrow \infty} \|\Delta_{X,u} \xi\|_{L^2,u}^2 = \|\Delta_{X,\infty} \xi\|_{L^2,\infty}^2 < \infty,$$

2. $\Delta_{X,\infty}$ is a linear operator from $A^{(0,0)}(X)$ to $\mathcal{H}_0(X)$.

3.

$$(\Delta_{X,\infty}\xi, \xi')_{L^2,\infty} = (\xi, \Delta_{X,\infty}\xi')_{L^2,\infty},$$

4.

$$(\Delta_{X,\infty}\xi, \xi)_{L^2,\infty} \geq 0,$$

for any $\xi, \xi' \in A^{0,0}(X)$.

Proof. Let $\xi \in A^{(0,0)}(X)$. We have for any $u > 1$:

$$\|\Delta_{X,u}\xi\|_{L^2,u}^2 = \int_{x \in X} (\Delta_{X,u}\xi, \Delta_{X,u}\xi)_x \omega_{X,u} = \frac{i}{2\pi} \int_{x \in X} \left(\Delta_{X,u}\xi, \Delta_{X,u}\xi \right)_x h_{X,u} \left(\frac{\partial}{\partial z}(x), \frac{\partial}{\partial \bar{z}}(x) \right)_x dz_x \wedge d\bar{z}_x,$$

where $\{\frac{\partial}{\partial z}(x)\}$ is local basis TX in a open subset U containing x .

For any $u \geq 1$, the Laplacian $\Delta_{X,u}$ has the following expression: $x \in U$:

$$\Delta_{X,u}\xi = -h \left(\frac{\partial}{\partial z}(x), \frac{\partial}{\partial \bar{z}}(x) \right)^{-1} \frac{\partial}{\partial z} \left(\frac{\partial \xi}{\partial \bar{z}_x} \right).$$

We have:

$$\begin{aligned} 0 &\leq h_{X,u} \left(\frac{\partial}{\partial z}(x), \frac{\partial}{\partial \bar{z}}(x) \right)_x \left(\Delta_{X,u}\xi, \Delta_{X,u}\xi \right)_x \\ &= h_{X,u} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \left(h_{X,u} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)^{-1} \frac{\partial}{\partial z} \left(\frac{\partial \xi}{\partial \bar{z}_x} \right), h_{X,u} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)^{-1} \frac{\partial}{\partial z} \left(\frac{\partial \xi}{\partial \bar{z}_x} \right) \right) \\ &= h_{X,u} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)_x^{-1} \frac{\partial}{\partial z} \left(\frac{\partial \xi}{\partial \bar{z}} \right) \frac{\partial}{\partial \bar{z}} \left(\frac{\partial \bar{\xi}}{\partial z} \right) \\ &\leq \frac{h_{X,\infty} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)_x}{h_{X,u} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)_x} h_{X,\infty} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)_x \left(\Delta_{X,\infty}\xi, \Delta_{X,\infty}\xi \right)_x, \end{aligned}$$

Note that $x \mapsto h_{X,\infty} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)_x h_{X,u} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)_x^{-1}$ is the restriction on U of a global continuous function on X , hence bounded. Recall that $(h_{X,u})_u \xrightarrow{u \rightarrow \infty} h_{X,\infty}$. Using a partition of unity and according to the dominated convergence theorem, we get:

$$\begin{aligned} \|\Delta_{X,\infty}\xi\|_{L^2,\infty}^2 &= \frac{i}{2\pi} \int_X h_{X,\infty} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)_x \left(\Delta_{X,\infty}\xi, \Delta_{X,\infty}\xi \right)_x dz \wedge d\bar{z} \\ &= \frac{i}{2\pi} \int_X h_{X,\infty} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)^{-1} \frac{\partial}{\partial z} \left(\frac{\partial \xi}{\partial \bar{z}} \right) \frac{\partial}{\partial \bar{z}} \left(\frac{\partial \bar{\xi}}{\partial z} \right) dz \wedge d\bar{z} \\ &= \frac{i}{2\pi} \int_X \lim_{u \rightarrow \infty} h_{X,u} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)^{-1} \frac{\partial}{\partial z} \left(\frac{\partial \xi}{\partial \bar{z}} \right) \frac{\partial}{\partial \bar{z}} \left(\frac{\partial \bar{\xi}}{\partial z} \right) dz \wedge d\bar{z} \\ &= \frac{i}{2\pi} \lim_{u \rightarrow \infty} \int_X h_{X,u} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)^{-1} \frac{\partial}{\partial z} \left(\frac{\partial \xi}{\partial \bar{z}} \right) \frac{\partial}{\partial \bar{z}} \left(\frac{\partial \bar{\xi}}{\partial z} \right) dz \wedge d\bar{z} \\ &= \lim_{u \rightarrow \infty} \|\Delta_{X,u}\xi\|_{L^2,u}^2, \end{aligned}$$

this proves the first claim.

The linearity in the second claim is obvious, and the second part follows from the first one. Now let $\xi, \xi' \in A^{(0,0)}(X)$, we have

$$\begin{aligned} (\Delta_{X,\infty}(\xi), \xi')_{L^2,\infty} &= \int_X \frac{\partial}{\partial z} \left(\frac{\partial \xi}{\partial \bar{z}} \right) \bar{\xi}' dz \wedge d\bar{z} \\ &= \int_X \frac{\partial \xi}{\partial \bar{z}} \frac{\partial \bar{\xi}'}{\partial z} dz \wedge d\bar{z}, \quad \text{by Stokes' theorem} \\ &= (\xi, \Delta_{X,\infty}(\xi'))_{L^2,\infty}. \end{aligned}$$

We infer,

$$(\Delta_{X,\infty} \xi, \xi)_{L^2,\infty} = \frac{i}{2\pi} \int_X \frac{\partial \xi}{\partial \bar{z}} \frac{\partial \bar{\xi}}{\partial z} dz \wedge d\bar{z} \geq 0.$$

□

We prove now, some technical results which will allow us to study the spectral properties of $\Delta_{X,\infty}$. We introduce the following function:

$$\delta_X(u) := \sup_{x \in X} \left| \frac{\partial}{\partial u} \left(\log h_{X,u} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)^{-1} \right) (x) \right| \quad \forall u > 1,$$

where $\{\frac{\partial}{\partial z}\}$ is a local holomorphic basis of TX .

Note that δ_X does not depend on the choice of the basis. Indeed, since $h_{X,u}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = (1 - \rho(u))h_{X,p-1}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) + \rho(u)h_{X,p}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$, for any $p \in \mathbb{N}^*$, $u \in [p-1, p]$ and $\{\frac{\partial}{\partial z}\}$ a local holomorphic basis of TX . Then

$$\begin{aligned} \frac{\partial}{\partial u} \log h_{X,u} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)^{-1} &= \rho(u) \frac{h_{X,p-1}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) - h_{X,p}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})}{h_{X,u}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})} \\ &= \rho(u) \frac{h_{X,p}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})}{h_{X,u}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})} \frac{h_{X,p-1}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) - h_{X,p}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})}{h_{X,p}(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})} \end{aligned}$$

Which is clearly a well defined continuous function on X . We know that $(h_{X,p})_{p \in \mathbb{N}}$ converges uniformly to $h_{X,\infty}$, then there is a constant c_1 such that:

$$\delta_X(u) \leq c_1 \left| \frac{h_{X,[u]} - h_{X,[u]+1}}{h_{X,[u]+1}} \right| \quad \forall u \geq 1, \quad (3)$$

where $[u]$ is the round down of u .

Proposition 3.4. *We have the following:*

$$\left\| \frac{\partial \Delta_{X,u}}{\partial u} \xi \right\|_{L^2,u} \leq \delta_X(u) \|\Delta_{X,u} \xi\|_{L^2,u},$$

for any $\xi \in A^{0,0}(X)$ and $u > 1$.

Proof. Fix $\xi \in A^{0,0}(X)$. Using the expression of the Laplacian, we have:

$$\frac{\partial \Delta_{X,u}}{\partial u} \xi = \frac{\partial}{\partial u} (\log h_{X,u} (\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})^{-1}) \Delta_{X,u} \xi \quad \forall \xi \in A^{(0,0)}(X).$$

Recall that $\frac{\partial}{\partial u} (\log h_{X,u} (\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})^{-1})$ is a continuous function on X which does not depend on the choice of the basis.

We have,

$$\begin{aligned} \left(\frac{\partial \Delta_{X,u}}{\partial u} \xi, \frac{\partial \Delta_{X,u}}{\partial u} \xi \right)_{L^2, u} &= \int_X \left| \frac{\partial}{\partial u} (\log h_{X,u} (\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})^{-1}) \right|^2 h_{X,u} (\Delta_{X,u} \xi, \Delta_{X,u} \xi) \omega_{X,u} \\ &\leq |\delta_X(u)|^2 \int_X h_{X,u} (\Delta_{X,u} \xi, \Delta_{X,u} \xi) \omega_{X,u} \\ &= |\delta_X(u)|^2 (\Delta_{X,u} \xi, \Delta_{X,u} \xi)_{L^2, u}. \end{aligned} \tag{4}$$

which yields the assertion. \square

Recall that for any $1 < u < \infty$, there exists a compact operator $(I + \Delta_{X,u})^{-1}$ on $\mathcal{H}_0(X)$ such that:

$$(I + \Delta_{X,u})^{-1} (I + \Delta_{X,u}) = I_0,$$

where I_0 means the identity operator of $A^{(0,0)}(X)$. We study next the variation of $(I + \Delta_{X,u})^{-1}$ with respect to u , but first let us recall the following fact: Let Δ be a Laplacian associated to smooth metrics. Then

$$\|(\Delta + I)^{-1}\| \leq 1, \tag{5}$$

where $\|\cdot\|$ is the induced metric. Indeed, we know that the eigenvectors of Δ form a complete orthonormal system for the completion of $A^{0,0}(X)$ with respect to the metrics. So, if we denote $(\phi_i)_i$ an orthonormal basis of eigenvectors of Δ , then for any $\xi \in A^{0,0}(X)$, there exists $(a_j)_{j \in \mathbb{N}}$ a sequence of complex numbers such that $\xi = \sum_i a_i \phi_i$ and one checks that

$$\|(\Delta + I)^{-1} \xi\|^2 = \left\| \sum_i \frac{a_i}{\lambda_i + 1} \phi_i \right\|^2 \leq \sum_i |a_i|^2 \|\phi_i\|^2 = \|\xi\|^2.$$

Proposition 3.5. *We have,*

$$\left\| \frac{\partial}{\partial u} (I + \Delta_{X,u})^{-1} \right\|_{L^2, \infty} \leq c_3 |\delta_X(u)| \quad \forall u > 1.$$

where c_3 is a constant.

Proof. We have,

$$\frac{\partial}{\partial u} (I + \Delta_{X,u})^{-1} = -(I + \Delta_{X,u})^{-1} \frac{\partial \Delta_{X,u}}{\partial u} (I + \Delta_{X,u})^{-1} \quad \forall u > 1.$$

Now let $\eta \in A^{(0,0)}(X)$ and put $\xi = (\Delta_{X,u} + I)^{-1} \eta$, we have

$$\begin{aligned} \left\| \frac{\partial \Delta_{X,u}}{\partial u} (\Delta_{X,u} + I)^{-1} \eta \right\|_{L^2, u}^2 &= \left\| \frac{\partial \Delta_{X,u}}{\partial u} \xi \right\|_{L^2, u}^2 \\ &\leq |\delta_X(u)|^2 (\Delta_{X,u} \xi, \Delta_{X,u} \xi)_{L^2, u} \quad \text{by (3.4)} \\ &= |\delta_X(u)|^2 (\Delta_{X,u} (\Delta_{X,u} + I)^{-1} \eta, \Delta_{X,u} (\Delta_{X,u} + I)^{-1} \eta)_{L^2, u} \\ &= |\delta_X(u)|^2 (\eta - (\Delta_{X,u} + I)^{-1} \eta, (\Delta_{X,u} + I)^{-1} \eta)_{L^2, u} \\ &\leq 2 |\delta_X(u)|^2 \|\eta\|_{L^2, u}^2, \end{aligned}$$

Thus

$$\left\| \frac{\partial}{\partial u} \Delta_{X,u} \cdot (\Delta_u + I)^{-1} \right\|_{L^2,u}^2 \leq 2|\delta_X(u)|^2 \quad \forall u > 1.$$

Since $\|\cdot\|_{L^2,\infty}$ and $\|\cdot\|_{L^2,u}$ for any $u > 1$ are uniformly equivalent, there exists c_2 a constant such that:

$$\left\| \frac{\partial}{\partial u} \Delta_{X,u} \cdot (\Delta_u + I)^{-1} \right\|_{L^2,\infty}^2 \leq c_2 |\delta_X(u)|^2 \quad \forall u > 1. \quad (6)$$

By the same argument and using the first claim, we can find a constant c_3 such that

$$\begin{aligned} \left\| \frac{\partial}{\partial u} (I + \Delta_{X,u})^{-1} \right\|_{L^2,\infty} &= \left\| (I + \Delta_{X,u})^{-1} \frac{\partial \Delta_{X,u}}{\partial u} (I + \Delta_{X,u})^{-1} \right\|_{L^2,\infty} \\ &\leq \|(\Delta_{X,u} + I)^{-1}\|_{L^2,\infty} \left\| \frac{\partial}{\partial u} \Delta_{X,u} \cdot (\Delta_{X,u} + I)^{-1} \right\|_{L^2,\infty} \\ &\leq c_3 |\delta_X(u)| \quad \forall u > 1. \end{aligned}$$

That is,

$$\left\| \frac{\partial}{\partial u} (I + \Delta_{X,u})^{-1} \right\|_{L^2,\infty} \leq c_3 |\delta_X(u)| \quad \forall u > 1.$$

□

Theorem 3.6. *The sequence $((I + \Delta_{X,u})^{-1})_{u \geq 1}$ converges to a compact operator denoted by $(I + \Delta_{X,\infty})^{-1} : \mathcal{H}_0(X) \rightarrow \mathcal{H}_0(X)$, with respect to $L^2_{X,\infty}$ -norm.*

Proof. According to (3), we may assume that $|\delta_X(u)| = O(\frac{1}{u^2})$. This yields for any $q > p$

$$\begin{aligned} \|(\Delta_{X,p} + I)^{-1} - (\Delta_{X,q} + I)^{-1}\|_{L^2,\infty} &= \left\| \int_p^q \frac{\partial}{\partial u} (I + \Delta_{X,u})^{-1} du \right\|_{L^2,\infty} \\ &\leq \int_p^q c_3 |\delta_X(u)| du \quad \text{by (3.5)} \\ &= \int_p^q O(\frac{1}{u^2}) du \\ &= O\left(\frac{1}{q} - \frac{1}{p}\right) \quad \forall p, q \gg 1. \end{aligned}$$

Therefore, the sequence of compact operators $((\Delta_{X,p} + I)^{-1})_{p \in \mathbb{N}}$ converges to a operator which we denote by $(I + \Delta_{X,\infty})^{-1}$. This operator is compact according to (4.4). □

3.2 A maximal positive selfadjoint extension of $\Delta_{X,\infty}$

The main goal of this paragraph is to prove that $\Delta_{X,\infty}$ admits a maximal selfadjoint extension to a large subspace denoted by $\mathcal{H}_2(X)$.

Let us begin first by reviewing some facts about the notion of selfadjoint extension of Laplacians. For simplicity, we restrict our selves to Compact Riemann surfaces. Let X be a compact Riemann surface. Let ω_X be normalized Kähler form on X , and $h_{\mathcal{O}}$ a constant metric on \mathcal{O} . For any $\phi, \psi \in A^{(0,0)}(X)$, we define an hermitian product (ϕ, ψ) as before, the corresponding norm will be denoted by $\|\cdot\|$, and we will call it the L^2 -norm. We let $\mathcal{H}_0(X)$ be the completion of the pre-Hilbert space $(A^{(0,0)}(X), (\cdot, \cdot))$. One proves $\mathcal{H}_0(X)$ does not depend on the choice of the metrics. More precisely, given two continuous

metrics on X and using the compactness of X , we obtain two equivalent metrics on $A^{(0,0)}(X)$.

When the metric of X is smooth, it is known that the Laplacian has a complete orthonormal sequence of smooth \mathcal{O} -valued eigenfunctions $\phi_0, \phi_1, \phi_2, \dots$ in $\mathcal{H}_0(X)$. In particular, we have:

$$\mathcal{H}_0(X) = \left\{ \phi = \sum_{k=0}^{\infty} a_k \phi_k \mid \|\phi\|^2 = \sum_{k=0}^{\infty} |a_k|^2 < \infty \right\}.$$

We set,

$$\mathcal{H}_2(X) = \left\{ \phi = \sum_{k=0}^{\infty} a_k \phi_k \mid \sum_{k=0}^{\infty} \lambda_k^2 |a_k|^2 < \infty \right\}.$$

We have,

$$A^{(0,0)}(X) \subseteq \mathcal{H}_2(X) \subseteq \mathcal{H}_0(X).$$

The right-hand side inclusion is obvious, and the other one can be deduced from [4, § 14.2.2 p.367]. Since $\mathcal{H}_0(X)$ is complete with respect to $\|\cdot\|$. Notice that $\mathcal{H}_2(X)$ is the completion of $A^{(0,0)}(X)$ with respect to the following norm $\|\cdot\|_2$, defined as follows: $\|\phi\|_2^2 = \|\phi\|^2 + \|\Delta\phi\|^2$, for any $\phi \in A^{(0,0)}(X)$.

One can view $\mathcal{H}_2(X)$ in a more intrinsic way, namely as the set of $\phi \in \mathcal{H}_0(X)$, such that there exists $(\phi_j)_{j \in \mathbb{N}}$, a sequence in $A^{(0,0)}(X)$ converging to ϕ with respect to L^2 -norm and such that the sequence $(\Delta\phi_j)_{j \in \mathbb{N}}$ admits a limit in $\mathcal{H}_0(X)$. So, we can write,

$$\mathcal{H}_2(X) = (I + \Delta)^{-1} \mathcal{H}_0(X).$$

Let $\phi \in \mathcal{H}(X)$, by definition there exists $(\phi_j)_{j \in \mathbb{N}}$ a sequence in $A^{(0,0)}(X)$ converging to ϕ and such that $(\Delta\phi_j)_{j \in \mathbb{N}}$ admits a limit. We can check easily that the limit is unique. This previous point motivates the following definition: We let Q be the operator on $\mathcal{H}_2(X)$ given by $Q(\phi) = \lim_{j \in \mathbb{N}} \Delta\phi_j$ for any $\phi \in \mathcal{H}_2(X)$ and $(\phi_j)_{j \in \mathbb{N}}$ described as above. Then Q is a maximal positive and selfadjoint extension of Δ with domain $\text{Dom}(Q) = \mathcal{H}_2(X)$. We claim that,

$$Q(\phi) = \psi - \phi, \tag{7}$$

for any $\phi \in \mathcal{H}_2(X)$, where ψ is the unique element in $\mathcal{H}_0(X)$ such that $\phi = (I + \Delta)^{-1} \psi$. Let us verify the last claim; Let $\phi \in \mathcal{H}_2(X)$, since $I + \Delta$ is invertible then there exists a unique $\psi \in \mathcal{H}_0(X)$ such that $\phi = (I + \Delta)^{-1} \psi$. Now, let $(\psi_j)_{j \in \mathbb{N}}$ be a sequence in $A^{(0,0)}(X)$ converging to ψ with respect to the L^2 -norm, it follows that

$$(\phi_j := (I + \Delta)^{-1} \psi_j)_{j \in \mathbb{N}}$$

converges to ϕ with respect to the L^2 -norm. Since, $Q(\phi_j) = \Delta\phi_j = \psi_j - \phi_j$ for any $j \in \mathbb{N}$, hence $(Q(\phi_j))_{j \in \mathbb{N}}$ converges to $\psi - \phi$. Then,

$$Q(\phi) = \psi - \phi,$$

If T is an extension of Δ , that is a linear selfadjoint operator $T : \text{Dom}(T) \rightarrow \mathcal{H}_0(X)$ such that: $\mathcal{H}_2(X) \subseteq \text{Dom}(T)$ and the restriction of T to $\mathcal{H}_2(X)$ is Q . Pick $\phi = \sum_{j=0}^{\infty} a_j \phi_j$ in $\mathcal{H}_0(X)$, then there exists $b_j \in \mathbb{C}$ for any $j \in \mathbb{N}$ such that:

$$T\phi = \sum_{j=0}^{\infty} b_j \phi_j,$$

We have,

$$b_j = (T(\phi), \phi_j) = (\phi, T(\phi_j)) = (\phi, Q(\phi_j)) = \lambda_j(\phi, \phi_j) = \lambda_j a_j.$$

Recall that $\|T\phi\|^2 = \sum_{j=0}^{\infty} |b_j|^2 < \infty$. From this we infer, $\sum_{j=0}^{\infty} \lambda_j^2 |a_j|^2 < \infty$, hence $\phi \in \mathcal{H}_2(X)$. We conclude that $T = Q$. We say Q is a maximal selfadjoint extension of Δ .

Our goal now is to construct Q_{∞} , a maximal selfadjoint extension of the operator $\Delta_{X,\infty}$. Let $(h_{X,u})_{u>1}$ be as before and let $(\Delta_{X,u})_{u>1}$ be the sequence of the associated Laplacians.

From (3.6), the sequence $((I + \Delta_{X,u})^{-1})_{u>1}$ converges to $(I + \Delta_{X,\infty})^{-1}$, with respect to one L^2_{X,u_0} -norm hence for any $L^2_{X,v}$ -norm with v fixed. We have also that $(I + \Delta_{X,\infty})^{-1}$ is a compact linear operator on $\mathcal{H}_0(X)$. Note that $(I + \Delta_{X,u})^{-1}\mathcal{H}_0(X)$ does not depend on u . Indeed, this follows from the previous discussion, and the fact that the metrics are uniformly equivalent. Then,

$$\mathcal{H}_2(X) = (I + \Delta_{X,\infty})^{-1}\mathcal{H}_0(X).$$

Claim 3.7. *Let H be a Hilbert space. Let $(\|\cdot\|_u)_{u\geq 1}$ be a sequence of uniformly equivalent hilbertian norms on H , converging to $\|\cdot\|_{\infty}$, a Hilbert norm on H .*

Let $(\eta_u)_{u\geq 1}$ and $(\eta'_u)_{u\geq 1}$ two sequences in H , converging respectively to η_{∞} and η'_{∞} with respect, hence any norm $\|\cdot\|_v$ with $v \geq 1$. Then, the complex sequence $((\eta_u, \eta'_u)_u)_{u\geq 1}$ converges to $(\eta_{\infty}, \eta'_{\infty})_{\infty}$.

Proof. We have,

$$(\eta_u, \eta'_u)_u - (\eta_{\infty}, \eta'_{\infty})_{\infty} = (\eta_u - \eta_{\infty}, \eta'_u)_u + (\eta_{\infty}, \eta'_u - \eta'_{\infty})_u + (\eta_{\infty}, \eta'_{\infty})_u - (\eta_{\infty}, \eta'_{\infty})_{\infty}.$$

Now, using the assumptions, there exists a constant M such that $|(\eta_u - \eta_{\infty}, \eta'_u)_u + (\eta_{\infty}, \eta'_u - \eta'_{\infty})_u| \leq M(\|\eta_u - \eta_{\infty}\|_u + \|\eta'_u - \eta'_{\infty}\|_u)$, and since $(\|\cdot\|_u)_{u\geq 1}$ converges to $\|\cdot\|_{\infty}$ we conclude that $((\eta_{\infty}, \eta'_{\infty})_u)_{u\geq 1}$ converges to $(\eta_{\infty}, \eta'_{\infty})_{\infty}$. \square

Let $\phi \in \mathcal{H}_2(X)$. There exists a $\psi \in \mathcal{H}_0(X)$ such that $\phi = (I + \Delta_{X,\infty})^{-1}\psi$. We claim that ψ is unique and we will prove the uniqueness later (see the proof of (3.8)). Then we define Q_{∞} , an extension of $\Delta_{X,\infty}$ as follows: Let $\phi \in \mathcal{H}_2$, so, by assumption, there exists a unique $\psi \in \mathcal{H}_0(X)$ such that $\phi = (I + \Delta_{X,\infty})^{-1}\psi$, we set:

$$Q_{\infty}(\phi) := \psi - \phi,$$

Let us check that Q_{∞} is a positive selfadjoint extension of $\Delta_{X,\infty}$. To establish the positivity of Q_{∞} we need the following claim: There exists a sequence $(\phi_u)_{u>1}$ in $\mathcal{H}_2(X)$ such that $(\phi_u)_{u>1}$ converges to ϕ with respect to any L^2 -norm, and such that $(Q_u(\phi_u))_{u>1}$ converges to $Q_{\infty}(\phi)$. Indeed, Let $\phi_u := (I + \Delta_{X,u})^{-1}\psi$, $\forall u > 1$. We have

$$((I + \Delta_{X,u})^{-1}\psi)_{u\geq 1} \xrightarrow{u \rightarrow \infty} (I + \Delta_{X,\infty})^{-1}\psi \quad \text{see (3.6)}$$

Note that $Q_u(\phi_u) = \psi - \phi_u$ (see (7)), which converges to $\psi - \phi = Q_{\infty}(\phi)$. Now, recall that Q_u is a positive operator with respect to $(\cdot, \cdot)_u$. Namely, $(Q_u(\phi_u), \phi_u)_u \geq 0$. Since $((\cdot, \cdot)_u)_{u\geq 1}$ converges uniformly to $(\cdot, \cdot)_{\infty}$ and according to the previous claim (3.7). We conclude that

$$(Q_{\infty}(\phi), \phi)_{\infty} \geq 0.$$

Using the same argument, we prove that Q_{∞} is selfadjoint.

Let $\phi \in A^{(0,0)}(X)$. By (3.3), the following element $\psi := (I + \Delta_{X,\infty})\phi$ belongs to $\mathcal{H}_0(X)$, and

$$(I + \Delta_{X,u})^{-1}\psi \xrightarrow{u \rightarrow \infty} (I + \Delta_{X,\infty})^{-1}\psi,$$

therefore,

$$Q_\infty(\phi) = \psi - \phi = \Delta_{X,\infty}\phi.$$

Let T be an extension of Q_∞ , that is a positive selfadjoint linear operator $T : \text{Dom}(T) \rightarrow \mathcal{H}_0(X)$ such that $\mathcal{H}_2(X) \subseteq \text{Dom}(T)$ and $T|_{\mathcal{H}_2(X)} = Q_\infty$. Let $\phi \in \text{Dom}(T)$, put $\psi := (I + T)\phi$. We have $\psi \in \mathcal{H}_0(X)$, thus $\theta := (I + \Delta_{X,\infty})^{-1}\psi \in \mathcal{H}_2(X)$. Hence,

$$(I + T)(\theta) = \theta + Q_\infty(\theta) = \theta + (\psi - \theta) = \psi.$$

But recall that $\psi = (I + T)\phi$, then

$$(T + I)(\theta - \phi) = 0.$$

Since T is a positive operator, and so is $T + I$, it follows that

$$\phi = \theta = (I + \Delta_{X,\infty})^{-1}\psi.$$

Therefore,

$$\text{Dom}(T) = \mathcal{H}_2(X) \quad \text{and} \quad T = Q_\infty.$$

So Q_∞ is a maximal positive selfadjoint extension for $\Delta_{X,\infty}$.

Theorem 3.8. *The operator $\Delta_{X,\infty}$ admits a maximal selfadjoint extension to $\mathcal{H}_2(X)$, we denote this extension also by $\Delta_{X,\infty}$. We have:*

$$(I + \Delta_{X,\infty})(I + \Delta_{X,\infty})^{-1} = I,$$

on $\mathcal{H}_0(X)$, where I is the operator identity of $\mathcal{H}_0(X)$.

$$(I + \Delta_{X,\infty})^{-1}(I + \Delta_{X,\infty}) = I,$$

on $\mathcal{H}_2(X)$, where I is the operator identity of $\mathcal{H}_2(X)$.

Proof. The first assertion follows from the previous discussion.

Recall that we supposed there exists a unique $\psi \in \mathcal{H}_0(X)$ such that $\phi = (I + \Delta_{X,\infty})^{-1}\psi$. Let us prove this point. It suffices to prove the following:

$$(I + \Delta_{X,\infty})(I + \Delta_{X,\infty})^{-1} = I,$$

on $\mathcal{H}_0(X)$.

We fix $\xi \in A^{(0,0)}(X)$. We have,

$$\lim_{u \rightarrow \infty} \|\Delta_{X,u}\xi\|_{L^2,u}^2 = \|\Delta_{X,\infty}\xi\|_{L^2,\infty}^2 < \infty \quad \text{see (3.3),}$$

and

$$\left\| \frac{\partial \Delta_{X,u}}{\partial u} \xi \right\|_{L^2,u} \leq \delta_X(u) \|\Delta_{X,u}\xi\|_{L^2,u} \quad \text{see (3.4).}$$

We deduce, (for fixed ξ), there exists C , a constant such that:

$$\left\| \frac{\partial \Delta_{X,u}}{\partial u} \xi \right\|_{L^2,u} \leq C \delta_X(u),$$

for $u \gg 1$. Remember that the different norms $L_{X,u}^2$ are uniformly equivalent, so we can find C' such that:

$$\left\| \frac{\partial \Delta_{X,u}}{\partial u} \xi \right\|_{L^2, \infty} \leq C' \delta_X(u),$$

Therefore,

$$\|\Delta_{X,p} \xi - \Delta_{X,q} \xi\|_{L^2, \infty} \leq C' \int_p^q \delta_X(u) du,$$

Thus $(\Delta_{X,p} \xi)_{p \in \mathbb{N}}$ converges to $\Delta_{X,\infty} \xi$ with respect to $L_{X,\infty}^2$.

Now, let $\psi \in \mathcal{H}_0(X)$ and $\xi \in A^{(0,0)}(X)$. Using (3.7), we have

$$\begin{aligned} ((\Delta_{X,\infty} + I)(\Delta_{X,\infty} + I)^{-1} \psi, \xi)_{L^2, \infty} &= ((\Delta_{X,\infty} + I)^{-1} \psi, (\Delta_{X,\infty} + I) \xi)_{L^2, \infty} \\ &= \lim_{u \rightarrow \infty} ((\Delta_{X,u} + I)^{-1} \psi, (\Delta_{X,u} + I) \xi)_{L^2, u} \\ &= \lim_{u \rightarrow \infty} (\psi, \xi)_{L^2, u} \\ &= (\psi, \xi)_{L^2, \infty}. \end{aligned}$$

So, we have proved that for any $\psi \in \mathcal{H}_0(X)$,

$$((\Delta_{X,\infty} + I)(\Delta_{X,\infty} + I)^{-1} \psi - \psi, \xi)_{L^2, \infty} = 0 \quad \forall \xi \in A^{(0,0)}(X).$$

To conclude, recall that if D is dense linear subspace of a Hilbert space $(H, (\cdot, \cdot)_H)$, and suppose there exists $v \in H$ such that $(v, z)_H = 0$ for all $z \in D$, so $v = 0$. Indeed, take $(z_j)_{j \in \mathbb{N}}$ a sequence in D converging to v . We have $(v, v)_H = \lim_{j \rightarrow \infty} (v, z_j)_H = 0$.

Applying this claim to $H = \mathcal{H}_0(X)$, $D = A^{(0,0)}(X)$ and $v = (\Delta_{X,\infty} + I)(\Delta_{X,\infty} + I)^{-1} \psi - \psi$. It follows that,

$$(\Delta_{X,\infty} + I)(\Delta_{X,\infty} + I)^{-1} = I.$$

on $\mathcal{H}_0(X)$.

Now, let us prove the last assertion of the theorem. Let $\xi \in \mathcal{H}_2(X)$ and $\psi \in \mathcal{H}_0(X)$, we have

$$\begin{aligned} ((\Delta_{X,\infty} + I)^{-1}(\Delta_{X,\infty} + I) \xi, \psi)_{L^2, \infty} &= ((\Delta_{X,\infty} + I) \xi, (\Delta_{X,\infty} + I)^{-1} \psi)_{L^2, \infty} \\ &= (\xi, (\Delta_{X,\infty} + I)(\Delta_{X,\infty} + I)^{-1} \psi)_{L^2, \infty} \\ &= (\xi, \psi)_{L^2, \infty}. \end{aligned}$$

Then,

$$(\Delta_{X,\infty} + I)^{-1}(\Delta_{X,\infty} + I) = I,$$

on $\mathcal{H}_0(X)$.

□

Corollary 3.9. $\Delta_{X,\infty}$ has an infinite positive discrete spectrum.

Proof. The existence of the spectrum and its nature follows, for instance, from [15, theorem 3.4 p.429]. The positivity is a consequence of the positivity of the operator $\Delta_{X,\infty}$. □

Theorem 3.10. $\Delta_{X,\infty}$ admit a heat kernel, we denote it by $e^{-t\Delta_{X,\infty}}$, $t > 0$.

Proof. We proved that $\Delta_{X,\infty}$ is a positive selfadjoint operator. Then from (4.13), we deduce that $\Delta_{X,\infty}$ generates a semi-group $e^{-t\Delta_{X,\infty}}$ for any $t > 0$. \square

Proposition 3.11. *There exists $(h_{X,u})_{u>1}$ a family of smooth hermitian metrics on TX converging uniformly to $h_{X,\infty}$, such that for any fixed $t > 0$:*

$$\left\| \frac{\partial}{\partial u} e^{-t\Delta_{X,u}} \right\|_{L^2,\infty} = O(\delta_X(u)) \quad \forall u \gg 1.$$

Proof. We have,

$$\begin{aligned} \frac{\partial}{\partial u} e^{-t\Delta^u} &= \int_0^t e^{-(t-s)\Delta^u} ((\partial_u \log h_{X,u}) \Delta^u) e^{-s\Delta^u} ds \quad \text{by (31)} \\ &= - \int_0^t e^{-(t-s)\Delta^u} (\partial_u \log h_{X,u}) \partial_s e^{-s\Delta^u} ds. \end{aligned}$$

For fixed $u > 0$, let $(\phi_{u,k})_{k \in \mathbb{N}}$ be an orthonormal basis with respect of $L^2_{X,u}$, as an example, we can choose a set of eigenvectors of $\Delta_{X,u}$. Let $\xi \in A^{(0,0)}(X)$, there exist real a_k for any $k \in \mathbb{N}$ such that $\xi = \sum_{k \in \mathbb{N}} a_{u,k} \phi_{u,k}$. We have,

$$\begin{aligned} \partial_t e^{-t\Delta_{X,u}} \xi &= - \sum_{k \in \mathbb{N}} a_{u,k} \lambda_{u,k} e^{-\lambda_{u,k} t} \phi_{u,k} \\ &= - \frac{1}{t} \sum_k a_{u,k} t \lambda_{u,k} e^{-\lambda_{u,k} t} \phi_{u,k}, \end{aligned}$$

Since $a^2 e^{-a} \leq 4e^{-2}, \forall a \geq 0$, it follows that

$$\begin{aligned} \left\| \partial_t e^{-t\Delta_{X,u}} \xi \right\|_{L^2,u}^2 &= \frac{1}{t^2} \sum_{k \in \mathbb{N}} a_{u,k}^2 (\lambda_{u,k} t)^2 e^{-2\lambda_{u,k} t} \\ &\leq \frac{e^{-2}}{t^2} \sum_{k \in \mathbb{N}} a_{u,k}^2 \\ &= \frac{e^{-2}}{t^2} \|\xi\|_{L^2,u}^2, \end{aligned}$$

Therefore,

$$\left\| \partial_t e^{-t\Delta_{X,u}} \right\|_{L^2,u} \leq \frac{e^{-1}}{t} \quad \forall t > 0.$$

Using the fact that all the norms are uniformly equivalent, we can find two constants M_1 and M_2 which are independent of u such that:

$$\left\| \partial_t e^{-t\Delta_{X,u}} \right\|_{L^2,\infty} \leq \frac{M_1}{t} \quad \forall u, \forall t > 0.$$

and

$$\left\| e^{-s\Delta_{X,u}} \right\|_{L^2,\infty} \leq M_2 \quad \forall s \geq 0.$$

Fix $t > 0$. for any $u \gg 1$

$$\begin{aligned}
\left\| \int_{\frac{t}{2}}^t e^{-(t-s)\Delta^u} ((\partial_u \log h_{X,u})) \partial_s e^{-s\Delta^u} ds \right\|_{L^2, \infty} &\leq \int_{\frac{t}{2}}^t \left\| e^{-(t-s)\Delta^u} ((\partial_u \log h_{X,u})) \partial_s e^{-s\Delta^u} \right\|_{L^2, \infty} ds \\
&\leq \int_{\frac{t}{2}}^t \delta_X(u) \|e^{-(t-s)\Delta_{X,u}}\|_{L^2, \infty} \|\partial_s e^{-s\Delta_{X,u}}\|_{L^2, \infty} ds \\
&\leq M_1 M_2 \delta_X(u) \int_{\frac{t}{2}}^t \frac{1}{s} ds \\
&= O(\delta_X(u))
\end{aligned}$$

Using an integration by parts, we get

$$\begin{aligned}
&\int_0^{\frac{t}{2}} e^{-(t-s)\Delta^u} ((\partial_u \log h_{X,u})) \partial_s e^{-s\Delta^u} ds \\
&= \left[e^{-(t-s)\Delta_{X,u}} (\partial_u \log h_{X,u}) e^{-s\Delta_{X,u}} \right]_0^{\frac{t}{2}} - \int_0^{\frac{t}{2}} \partial_s (e^{-(t-s)\Delta_{X,u}}) (\partial_u \log h_{X,u}) e^{-s\Delta_{X,u}} ds \\
&= e^{-\frac{t}{2}\Delta_{X,u}} (\partial_u \log h_{X,u}) e^{-\frac{t}{2}\Delta_{X,u}} - e^{-t\Delta_{X,u}} (\partial_u \log h_{X,u}) I - \int_0^{\frac{t}{2}} \partial_s (\mathcal{O}^{-(t-s)\Delta^u}) (\partial_u \log h_{X,u}) e^{-s\Delta^u} ds \\
&= e^{-\frac{t}{2}\Delta_{X,u}} (\partial_u \log h_{X,u}) e^{-\frac{t}{2}\Delta_{X,u}} - e^{-t\Delta_{X,u}} (\partial_u \log h_{X,u}) I + \int_{\frac{t}{2}}^t \partial_s (\mathcal{O}^{-s\Delta^u}) (\partial_u \log h_{X,u}) e^{-(t-s)\Delta^u} ds,
\end{aligned}$$

Then there exists a constant M_3 such that:

$$\left\| \int_0^{\frac{t}{2}} e^{-(t-s)\Delta^u} ((\partial_u \log h_{X,u})) \partial_s e^{-s\Delta^u} ds \right\|_{L^2, \infty} \leq M_3 \delta_X(u).$$

We conclude that:

$$\left\| \frac{\partial}{\partial u} e^{-t\Delta_{X,u}} \right\|_{L^2, \infty} = O(\delta_X(u)) \quad \forall u \gg 1. \quad (8)$$

□

As an application of the previous results, we show that $e^{-t\Delta_{X,\infty}}$ can be approximate by a sequence of heat kernels associated to smooth metrics.

Theorem 3.12. *for any $t > 0$, we have:*

$$(e^{-t\Delta_{X,u}})_u \xrightarrow{u \rightarrow +\infty} e^{-t\Delta_{X,\infty}},$$

In particular, $e^{-t\Delta_{X,\infty}}$ is a compact operator from $\mathcal{H}_0(X)$ to $\mathcal{H}_0(X)$.

Proof. By (8) and (4.4), the sequence $(e^{-t\Delta_{X,u}})_{u>1}$ converges to a limit, which will be denoted by L_t , for any $t > 0$. According to (4.4), L_t is a compact operator.

By assumption, we may assume that $1 - \frac{h_{X,\infty}}{h_{X,u}} = O(\frac{1}{u})$ for any $u \gg 1$. On the other hand, we have $\Delta_{X,\infty} = \frac{h_{X,\infty}}{h_{X,u}} \Delta_{X,u}$. Then

$$\begin{aligned} (\partial_t + \Delta_{X,\infty})e^{-t\Delta_{X,u}} &= (\partial_t + \frac{h_{X,\infty}}{h_{X,u}}\Delta_{X,u})e^{-t\Delta_{X,u}} \\ &= \left(1 - \frac{h_{X,\infty}}{h_{X,u}}\right) \partial_t e^{-t\Delta_{X,u}} \\ &= O\left(\frac{1}{u}\right) \partial_t e^{-t\Delta_{X,u}}. \end{aligned}$$

If we fix $t_0 > 0$, we have shown that $\|\partial_t e^{-t\Delta_{X,u}}\|_{L^2,\infty}$ is bounded for any $t \geq t_0$. It follows that

$$(\partial_t + \Delta_{X,\infty})L_t = 0 \quad \forall t \geq t_0.$$

Moreover $L_t \rightarrow I$ when $t \mapsto 0$. Since $e^{-t\Delta_{X,\infty}}$ satisfies the same properties, and by the uniqueness of the heat kernel solution, it follows that $L_t = e^{-t\Delta_{X,\infty}}$, for any $t > 0$. \square

3.3 The trace and the Zeta function associated to $\Delta_{X,\infty}$

For any $u \in [1, \infty]$, we consider the norm $L_{X,u}^2$ on $A^{(0,0)}(X)$, we recall that this norm is induced by $h_{X,u}$ and $h_{\mathcal{O}}$ (a constant metric on \mathcal{O}). An operator T on the completion of $A^{(0,0)}(X)$ with respect to $L_{X,u}^2$ is said to be of trace-class if the sum $\sum_{k \in \mathbb{N}} (T\xi_{u,k}, \xi_{u,k})_{L^2,u}$, is absolutely convergent for one, hence any orthonormal basis $(\xi_{u,k})_{k \in \mathbb{N}}$. The value of this sum, which is independent of the choice of the basis, is called the trace of T , and will be denoted here by $\text{Tr}_u(T)$.

For any $u \in [1, \infty[$, we consider the operator $P^u e^{-t\Delta_{X,u}}$, where P^u is the orthogonal projection with kernel equal to $H^0(X, \mathcal{O})$, with respect to $L_{X,u}^2$. We will need the following lemma, which describes the variation of $(P^u)_{u \geq 1}$ with respect to u sufficiently large:

Lemma 3.13. *We have, P^u is a bounded operator and*

$$\frac{\partial P^u}{\partial u} = O(\delta_X(u)), \quad \forall u \gg 1.$$

Proof. Let $1 < u < \infty$. By definition of P^u , we have for any $\xi \in A^{0,0}(X)$, there exists $a^{(u)}(\xi)$ a complex number, such that

$$P^u \xi = \xi + a^{(u)}(\xi)^2.$$

Then $a^{(u)}(\xi)$ satisfies:

$$a^{(u)}(\xi) = -\frac{(\xi, 1)_{L^2,u}}{(1, 1)_{L^2,u}}.$$

We have

$$\frac{\partial P^{(u)}}{\partial u} \xi = \frac{\partial a^{(u)}(\xi)}{\partial u} = -(1, 1)_{L^2,u}^{-1} \frac{\partial}{\partial u} (\xi, 1)_{L^2,u} + (\xi, 1)_{L^2,u} (1, 1)_{L^2,u}^{-2} \frac{\partial}{\partial u} (1, 1)_{L^2,u}.$$

Since $(h_{X,u})_u$ converges to $h_{X,\infty}$ uniformly, then $\left((1, 1)_{L^2,u}^{-1}\right)_u$ is bounded. Then, there exist constants $m, m' > 0$ such that:

$$\|P^u \xi\|_{L^2,\infty} \leq \|\xi\|_{L^2,\infty} + m' \|\xi\|_{L^2,u},$$

²We view $a^{(u)}(\xi)$ as an element in $A^{(0,0)}(X)$.

then P^u is bounded with respect to $L_{X,\infty}^2$, and

$$\left\| \frac{\partial P^{(u)}}{\partial u} \xi \right\|_{L^2,\infty} \leq m \left\| \frac{\partial}{\partial u} (\xi, 1)_{L^2,u} \right\|_{L^2,\infty} + m \|\xi\|_{L^2,u} \left\| \frac{\partial}{\partial u} (1, 1)_{L^2,u} \right\|_{L^2,u} \quad \forall u > 1.$$

Let $\xi, \eta \in A^{(0,0)}(X)$, we have

$$\begin{aligned} \left| \frac{\partial}{\partial u} (\xi, \eta)_{L^2,u} \right| &= \left| \int_X \xi \bar{\eta} \frac{\partial}{\partial u} (\omega_{X,u}) \right| \\ &= \left| \int_X \xi \bar{\eta} \left(\frac{\partial}{\partial u} \log h_{X,u} \right) \omega_{X,u} \right| \\ &= \left| \left(\xi, \left(\frac{\partial}{\partial u} \log h_{X,u} \right) \eta \right)_{L^2,u} \right| \\ &\leq \delta_X(u) \|\xi\|_{L^2,u} \|\eta\|_{L^2,u} \quad \text{by Cauchy-Schwartz inequality.} \end{aligned}$$

Recall that the metrics are uniformly equivalent, we deduce there exists a constant m'' such that:

$$\left\| \frac{\partial P^{(u)}}{\partial u} \xi \right\|_{L^2,\infty} \leq m'' \delta_X(u) \|\xi\|_{L^2,\infty} \quad \forall u > 1.$$

Then,

$$\frac{\partial P^u}{\partial u} = O(\delta_X(u)), \quad u \gg 1.$$

□

We need also this technical lemma:

Lemma 3.14. *Let $\{c_{n,i} : n \in \mathbb{N}, i \in N\}$ be a family of positive real. We have*

$$\liminf_{n \rightarrow \infty} \sum_i c_{n,i} \geq \sum_i \liminf_{n \rightarrow \infty} c_{n,i}.$$

Proof. Let N be an nonzero integer. We have

$$\sum_{i=1}^{\infty} c_{k,i} \geq \sum_{i=1}^N \inf_{l \geq n} c_{l,i}, \quad \forall n \quad \forall k \geq n$$

then,

$$\inf_{k \geq n} \sum_{i=1}^{\infty} c_{k,i} \geq \sum_{i=1}^N \inf_{l \geq n} c_{l,i}, \quad \forall n$$

then,

$$\liminf_n \sum_i c_{n,i} \geq \sum_{i=1}^N \liminf_n c_{n,i}.$$

Since all the terms are positive, we conclude that,

$$\liminf_n \sum_i c_{n,i} \geq \sum_i \liminf_n c_{n,i}.$$

□

We recall that when $u \geq 1$, the metric $h_{X,u}$ is smooth, then the spectral theory of generalized Laplacians states that the operator $P^u e^{-t\Delta_{X,u}}$ is of trace-class, for any $t > 0$. One defines the so-called Theta function given by $\theta_{X,u}(t) = \text{Tr}_u(P^u e^{-t\Delta_{X,u}})$. If $(\lambda_{u,k})_{k \in \mathbb{N}}$ is the set of eigenvalues of $\Delta_{X,u}$ counted with multiplicity and in increasing order, we have

$$\theta_{X,u}(t) = \sum_{k \in \mathbb{N}^*} e^{-t\lambda_{u,k}} \quad \forall t > 0,$$

we refer the reader to chapter 2 of [2].

The smoothness of the metrics is a necessary condition in order to use the classical spectral theory of Laplacians. But, an integrable metric may be singular, as in following example: Let \mathbb{P}^1 the complex projective line. Since $T\mathbb{P}^1 \simeq \mathcal{O}(2)$, we can endow \mathbb{P}^1 with the following metric:

$$h_{\mathbb{P}^1,\infty}(s,s)([x_0 : x_1]) := \frac{|s([x_0, x_1])|^2}{\max(|x_0|, |x_1|)^4}, \quad \forall [x_0, x_1] \in \mathbb{P}^1$$

where s is a local holomorphic section of $\mathcal{O}(2)$. Then, we can show that $h_{\mathbb{P}^1,\infty}$ is a singular integrable metric.

Definition 3.15. We set,

$$\theta_{X,\infty}(t) := \text{Tr}(P^\infty e^{-t\Delta_{X,\infty}}) \quad \forall t > 0,$$

where P^∞ is the orthogonal projection with respect to $L_{X,\infty}^2$ with $H^0(X, \mathcal{O})$ as kernel. We say $\theta_{X,\infty}$ is the Theta function associated to $\Delta_{X,\infty}$.

Theorem 3.16. We have,

$$0 \leq \theta_{X,\infty}(t) < \infty \quad \forall t > 0,$$

hence $P^\infty e^{-t\Delta_{X,\infty}}$ is a trace-class operator for any $t > 0$.

Proof. From (26),

$$\left| \text{Tr}_\infty((\Delta_{X,u} + I)^{-2}) \right| \leq \|(\Delta_{X,u} + I)^{-2}\|_{1,\infty} \quad \forall u \geq 1. \quad (9)$$

and by (4.11), we have for any $0 < \varepsilon \ll 1$ and any $u \gg 1$:

$$\frac{1-\varepsilon}{1+\varepsilon} \|(\Delta_{X,u} + I)^{-2}\|_{1,u} \leq \|(\Delta_{X,u} + I)^{-2}\|_{1,\infty} \leq \frac{1+\varepsilon}{1-\varepsilon} \|(\Delta_{X,u} + I)^{-2}\|_{1,u}. \quad (10)$$

Also, we have:

$$\|(\Delta_{X,u} + I)^{-2}\|_{1,u} = \sum_{k \in \mathbb{N}} \frac{1}{(\lambda_{u,k} + 1)^2} \leq \zeta_{X,u}(2) + 1 < \infty \quad \forall u \geq 1, \quad (11)$$

where $(\lambda_{u,k})_{k \in \mathbb{N}}$ is the set of eigenvalues of $\Delta_{X,u}$, ordered in increasing order, and $\zeta_{X,u}$ is the Zeta function attached to $\Delta_{X,u}$ which is finite on the set $\{s \in \mathbb{C} \mid \text{Re}(s) > 1\}$, see [2, § 9.6].

From (9), (10) and (11), we conclude that $(\Delta_{X,u} + I)^{-2}$ is a trace-class operator for the norm $L_{X,\infty}^2$, for any u sufficiently large. Hence $\text{Tr}((\Delta_{X,u} + I)^{-2})$ is finite.

By the Lidskii theorem, (cf. (4.8)) and since $(\Delta_{X,u} + I)^{-2}$ is a trace-class operator for $L_{X,\infty}^2$, then $\text{Tr}_\infty((\Delta_{X,u} + I)^{-2})$ is the sum of its eigenvalues counted with their multiplicity. But they are positive, hence

$$\text{Tr}_\infty((\Delta_{X,u} + I)^{-2}) = \|(\Delta_{X,u} + I)^{-2}\|_{1,u}.$$

We have

$$\begin{aligned}\frac{\partial}{\partial u}(\Delta_{X,u} + I)^{-2} &= -(\Delta_{X,u} + I)^{-2} \frac{\partial}{\partial u}(\Delta_{X,u} + I)^2 (\Delta_{X,u} + I)^{-2} \\ &= -(\Delta_{X,u} + I)^{-2} \frac{\partial \Delta_{X,u}}{\partial u} (\Delta_{X,u} + I)^{-1} - (\Delta_{X,u} + I)^{-1} \frac{\partial \Delta_{X,u}}{\partial u} (\Delta_{X,u} + I)^{-2},\end{aligned}$$

hence, for u sufficiently large

$$\begin{aligned}\left| \frac{\partial}{\partial u} \text{Tr}_\infty \left((\Delta_{X,u} + I)^{-2} \right) \right| &= \left| \text{Tr}_\infty \left((\Delta_{X,u} + I) \frac{\partial \Delta_{X,u}}{\partial u} (\Delta_{X,u} + I)^{-1} \right) + \text{Tr}_\infty \left((\Delta_{X,u} + I)^{-1} \frac{\partial \Delta_{X,u}}{\partial u} (\Delta_{X,u} + I) \right) \right| \\ &= 2 \left| \text{Tr}_\infty \left((\Delta_{X,u} + I) \frac{\partial \Delta_{X,u}}{\partial u} (\Delta_{X,u} + I)^{-1} \right) \right| \quad \text{by (25)} \\ &\leq 2 \left\| (\Delta_{X,u} + I) \frac{\partial \Delta_{X,u}}{\partial u} (\Delta_{X,u} + I)^{-1} \right\|_{1,\infty} \quad \text{by (26)} \\ &\leq c_2 \delta_X(u) \left\| (\Delta_{X,u} + I)^{-2} \right\|_{1,\infty} \quad \text{by (6) and (4.9)} \\ &\leq c \delta_X(u) \left\| (\Delta_{X,u} + I)^{-2} \right\|_{1,u} \quad \text{the existence of } c \text{ follows from (10).}\end{aligned}$$

We have proved then the following:

$$\left| \frac{\partial}{\partial u} \text{Tr}_\infty \left((\Delta_{X,u} + I)^{-2} \right) \right| \leq c \delta_X(u) \text{Tr}_\infty \left((\Delta_{X,u} + I)^{-2} \right).$$

If we set $\alpha(u) := \text{Tr}_\infty \left((\Delta_{X,u} + I)^{-2} \right)$, then the last inequality becomes:

$$\left| \frac{\partial}{\partial u} \alpha(u) \right| \leq c \delta_X(u) \alpha(u) \quad \forall u > 1.$$

Thus,

$$\left| \log \left(\frac{\alpha(u)}{\alpha(u')} \right) \right| \leq c \left| \int_u^{u'} \delta_X(v) dv \right| \quad \forall u > 1.$$

From (3), we can choose $(h_{X,u})_{u \geq 1}$ such that : $\delta_X(u) = O(\frac{1}{u^2})$ for any $u \gg 1$. Then,

$$\left| \log \left(\frac{\alpha(u)}{\alpha(u')} \right) \right| \leq \left| \int_u^{u'} O\left(\frac{1}{v^2}\right) dv \right| = O\left(\left|\frac{1}{u} - \frac{1}{u'}\right|\right) \quad \forall u, u' \gg 1,$$

It follows that $u \mapsto \alpha(u)$ is bounded on a interval of the form $[A, \infty[$, and we choose $A > 1$.

For any $t > 0$, there exists c_t a constant such that:

$$e^{-ta} \leq \frac{c_t}{(1+a)^2} \quad \forall a > 0.$$

If we denote by $\theta_{X,u}$ the Theta function associated to $\Delta_{X,u}$, then

$$\theta_{X,u}(t) = \sum_{k=1}^{\infty} e^{-t\lambda_{u,k}} \leq c_t \sum_{k=0}^{\infty} \frac{1}{(\lambda_{u,k} + 1)^2} = c_t \alpha(u) \quad \forall u \geq 1.$$

In particular, this inequality holds for any $u \geq A$. Since, $u \mapsto \alpha(u)$ is bounded on $[A, \infty[$, we conclude that for any $t > 0$ fixed, the following sequence:

$$(\theta_{X,u}(t))_{u \geq A},$$

is bounded.

To establish the theorem, that is $P^\infty e^{-t\Delta_{X,\infty}}$ is a trace-class operator for any $t > 0$, it suffices to find $0 < \varepsilon \ll 1$ such that:

$$\theta_{X,\infty}(t) \leq \frac{1+\varepsilon}{1-\varepsilon} \liminf_{u \rightarrow \infty} \theta_{X,u}(t) \quad \forall t > 1,$$

and since the right-hand side is bounded, this yields to:

$$\theta_{X,\infty}(t) < \infty \quad \forall t > 0.$$

So it remains to prove there exists $0 < \varepsilon \ll 1$ such that:

$$\theta_{X,\infty}(t) \leq \frac{1+\varepsilon}{1-\varepsilon} \liminf_{u \rightarrow \infty} \theta_{X,u}(t) \quad \forall t > 1.$$

In order to prove this claim, we start by comparing $\sigma_n(P^\infty e^{-t\Delta_{X,\infty}})_\infty^3$ with $\sigma_n(P^u e^{-t\Delta_{X,u}})_\infty$ for any $u \gg 1$. Let R be a finite-rank operator such that its rank $\leq n$. We fix $t > 0$, we have

$$\begin{aligned} \sigma_n(P^\infty e^{-t\Delta_{X,\infty}})_\infty &\leq \|P^\infty e^{-t\Delta_{X,\infty}} - R\|_{L^2,\infty} \\ &\leq \|P^\infty e^{-t\Delta_{X,\infty}} - P^\infty e^{-t\Delta_{X,u}}\|_{L^2,\infty} + \|P^\infty e^{-t\Delta_{X,u}} - P^u e^{-t\Delta_{X,u}}\|_{L^2,\infty} + \|P^u e^{-t\Delta_{X,u}} - R\|_{L^2,\infty} \\ &\leq \|P^\infty\|_{L^2} \|e^{-t\Delta_{X,\infty}} - e^{-t\Delta_{X,u}}\|_{L^2,\infty} + \|P^\infty - P^u\|_{L^2,\infty} \|e^{-t\Delta_{X,u}}\|_{L^2,\infty} + \|P^u e^{-t\Delta_{X,u}} - R\|_{L^2,\infty}. \end{aligned}$$

Since the last inequality holds for an arbitrary R , then

$$\sigma_n(P^\infty e^{-t\Delta_{X,\infty}})_\infty \leq \|P^\infty\|_{L^2} \|e^{-t\Delta_{X,\infty}} - e^{-t\Delta_{X,u}}\|_{L^2,\infty} + \|P^\infty - P^u\|_{L^2,\infty} \|e^{-t\Delta_{X,u}}\|_{L^2,\infty} + \sigma_n(P^u e^{-t\Delta_{X,u}})_\infty.$$

According to (3.12) and (3.13), we know that $(\mathcal{O}^{-t\Delta_{X,u}})_{u \geq 1}$ (resp. $(P^u)_{u \geq 1}$) converges to $e^{-t\Delta_{X,\infty}}$ (resp. to P^∞) with respect to the norm $L_{X,\infty}^2$, and that the real sequence $(\|e^{-t\Delta_{X,u}}\|_{L^2,\infty})_u$ is bounded, hence the previous inequality yields to:

$$\sigma_n(P^\infty e^{-t\Delta_{X,\infty}})_\infty \leq \liminf_{u \rightarrow \infty} \sigma_n(P^u e^{-t\Delta_{X,u}})_\infty. \quad (12)$$

Therefore,

$$\begin{aligned} \theta_{X,\infty}(t) &= \text{Tr}_\infty(P^\infty e^{-t\Delta_{X,\infty}}) \\ &= \sum_{n \in \mathbb{N}} \sigma_n(P^\infty e^{-t\Delta_{X,\infty}})_\infty \quad \text{see the definition (4.5)} \\ &\leq \sum_{n \in \mathbb{N}} \liminf_{u \rightarrow \infty} \sigma_n(P^u e^{-t\Delta_{X,u}})_\infty \quad \text{by (12)} \\ &\leq \liminf_{u \rightarrow \infty} \sum_{n \in \mathbb{N}} \sigma_n(P^u e^{-t\Delta_{X,u}})_\infty \quad \text{by (3.14)} \\ &= \liminf_{u \rightarrow \infty} \|P^u e^{-t\Delta_{X,u}}\|_{1,\infty} \quad \text{see the definition (4.5)}. \end{aligned}$$

So,

$$\theta_{X,\infty}(t) \leq \liminf_{u \rightarrow \infty} \|P^u e^{-t\Delta_{X,u}}\|_{1,\infty}. \quad (13)$$

Let $0 < \varepsilon < 1$, using (4.11), we get

$$\frac{1-\varepsilon}{1+\varepsilon} \|P^u e^{-t\Delta_{X,u}}\|_{1,\infty} \leq \|P^u e^{-t\Delta_{X,u}}\|_{1,u} \leq \frac{1+\varepsilon}{1-\varepsilon} \|P^u e^{-t\Delta_{X,u}}\|_{1,\infty} \quad \forall u \gg 1. \quad (14)$$

³See (24) for the definition of $\sigma_n(\cdot)$

Recall that

$$\theta_{X,u}(t) = \|P^u e^{-t\Delta_{X,u}}\|_{1,u}. \quad (15)$$

Finally, taking into account (13), (14) and (15) we obtain:

$$\theta_{X,\infty}(t) \leq \frac{1+\varepsilon}{1-\varepsilon} \liminf_{u \rightarrow \infty} \theta_{X,u}(t).$$

We conclude that for any $t > 0$, the operator $P^\infty e^{-t\Delta_{X,\infty}}$ is of trace-class. \square

We denote by $(\lambda_{\infty,k})_{k \in \mathbb{N}}$ the sequence of the eigenvalues of $\Delta_{X,\infty}$ counted with their multiplicity, and ordered in increasing order.

Theorem 3.17. *For any $t > 0$ fixed, we have:*

$$(\theta_{X,u}(t))_{u \geq 1} \xrightarrow{u \rightarrow \infty} \theta_{X,\infty}(t),$$

and,

$$\zeta_{X,\infty}(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \theta_{X,\infty}(t) dt = \sum_{k=1}^\infty \frac{1}{\lambda_{\infty,k}^s},$$

is finite for any $s \in \mathbb{C}$, such that $\operatorname{Re}(s) > 1$. This function of s admits a meromorphic continuation to the whole complex plane with a pole at $s = 1$, and it is holomorphic at $s = 0$. We have,

$$\zeta'_{X,\infty}(0) = \int_1^\infty \frac{\theta_{X,\infty}(t)}{t} dt + \gamma b_{\infty,-1} - b_{\infty,0} + \int_0^1 \frac{\rho_{X,\infty}(t)}{t} dt = \lim_{u \rightarrow \infty} (\zeta'_{X,u}(0))_{u \geq 1},$$

where $b_{\infty,-1}, b_{\infty,0}$ are real numbers and $\rho_{X,\infty}$ is a real function such that $\theta_{X,\infty}(t) = \frac{b_{\infty,-1}}{t} + b_{\infty,0} + \rho_{X,\infty}(t)$, and $\rho_{X,\infty}(t) = O(t)$ for $t > 0$ sufficiently small.

We introduce then the following definition:

Definition 3.18. Let X be a compact Riemann surface endowed with a integrable metric, $h_{X,\infty}$. We set

$$T((X, \omega_{X,\infty}), (\mathcal{O}, h_{\mathcal{O}})) := \zeta'_{\Delta_{X,\infty}}(0),$$

and we call it the holomorphic analytic torsion attached to $((X, \omega_{X,\infty}), (\mathcal{O}, h_{\mathcal{O}}))$, where $h_{\mathcal{O}}$ is a constant metric.

Before to prove this theorem, recall that for any $u > 1$ and any integer k , there exist real numbers $a_{u,-1}, a_{u,0}, \dots, a_{u,k}$ such that:

$$\theta_{X,u}(t) = \sum_{i=-1}^k a_{u,i} t^i + O(t^{k+1}),$$

for any t small enough, see [2, p. 94]. Notice that, $a_{u,-1} = 4\pi \operatorname{vol}_u(X)$, see [2, theorem 2.41].

Proposition 3.19. *We consider $(h_{X,u})_{u>1}$ as before. Then $(a_{u,0})_{u \geq 1}$ and $(a_{u,-1})_{u \geq 1}$ converge respectively to finite limit when u goes to ∞ .*

Proof. We know that:

$$a_{u,-1} = 4\pi \operatorname{vol}_u(X).$$

Since, the sequence of hermitian norms, $(L_{X,u}^2)_{u>0}$ tends to $L_{X,\infty}^2$. It follows that $\lim_{u \rightarrow \infty} a_{u,-1} = 4\pi \operatorname{rg}(\mathcal{O}) \operatorname{vol}_\infty(X) =: a_{\infty,-1}$.

Let us prove the second assertion. Let $u > 1$, and t be a positive real, and we consider $((TX, th_{X,u}); (\mathcal{O}, h_{\mathcal{O}}))$. The variation of Quillen metrics associated to t is given by the following anomaly formula, see [3]:

$$-\log h_{Q,((TX, th_{X,u}); (\mathcal{O}, h_{\mathcal{O}}))} + \log h_{Q,((TX, h_{X,u}); (\mathcal{O}, h_{\mathcal{O}}))} = \int_X ch(\mathcal{O}, h_{\mathcal{O}}) \widetilde{Td}(TX, th_{X,u}, h_{X,u}),$$

One checks, using the local expression of the Laplacian, that:

$$\Delta_{t,u} := \Delta_{((TX, th_{X,u}); (\mathcal{O}, h_{\mathcal{O}}))} = t^{-1} \Delta_{((TX, h_{X,u}); (\mathcal{O}, h_{\mathcal{O}}))} = t^{-1} \Delta_{1,u} \quad \forall t > 0.$$

It follows that $\zeta'_{\Delta_{t,u}}(0) = \zeta_{\Delta_{1,u}}(0) \log t + \zeta'_{\Delta_{1,u}}(0)$, where $\zeta_{\Delta_{t,u}}$ denote the Zeta function associated to the data $((TX, th_{X,u}); (\mathcal{O}, h_{\mathcal{O}}))$. We verify that:

$$\widetilde{Td}(TX, th_{X,u}, h_{X,u}) = \frac{1}{2} \log t + \frac{1}{6} (\log t) c_1(TX, h_{X,u}),$$

in $\oplus_{p \geq 0} \widetilde{A}^{(p,p)}(X)$, see [7] for the definition of Bott-Chern classes. Since $\text{Vol}_{th_{X,u}} = t^{\dim X} \text{Vol}_{h_{X,u}}$, then

$$h_{L^2,((TX, th_{X,u}); (\mathcal{O}, h_{\mathcal{O}}))} = t^{2 \dim H^0(X, \mathcal{O})} h_{L^2,((TX, h_{X,u}); (\mathcal{O}, h_{\mathcal{O}}))}.$$

Recall that,

$$h_{Q,((TX, h_{X,u}); (\mathcal{O}, h_{\mathcal{O}}))} = h_{L^2,((TX, h_{X,u}); (\mathcal{O}, h_{\mathcal{O}}))} \exp(-\zeta'_{\Delta_{1,u}}(0)).$$

Using the previous anomaly formula, we get

$$-2 \dim H^0(X, \mathcal{O}) \log t + \zeta_{\Delta_{1,u}}(0) \log t = \frac{1}{2} \log t \int_X c_1(\mathcal{O}, h_{\mathcal{O}}) + \frac{1}{6} \log t \int_X c_1(TX, h_X).$$

Remember that $\zeta_{\Delta_{X,u}}(0) = a_{u,0}$ (see for instance [19, theorem.1]), hence:

$$a_{u,0} = \frac{1}{6} \int_X c_1(TX) + 2 \quad \forall u > 1. \quad (16)$$

Therefore, $a_{u,0}$ does not depend on u . □

Proof. In order to study $\theta_{X,\infty}$ we introduce for any $u \geq 1$, the following auxiliary function:

$$\theta_{u,\infty}(t) := \|P^u e^{-t\Delta_{X,u}}\|_{1,\infty} \quad \forall t > 1.$$

Note that this function is finite for any $t > 0$, this follows easily from (14). We have for any $u, u' > 1$,

$$\begin{aligned} |\theta_{u,\infty}(t) - \theta_{u',\infty}(t)| &= \left| \|P^u e^{-t\Delta_{X,u}}\|_{1,\infty} - \|P^{u'} e^{-t\Delta_{X,u'}}\|_{1,\infty} \right| \\ &\leq \|P^u e^{-t\Delta_{X,u}} - P^{u'} e^{-t\Delta_{X,u'}}\|_{1,\infty} \\ &= \left\| \int_u^{u'} \frac{\partial}{\partial v} (P^v e^{-t\Delta_{X,v}}) dv \right\|_{1,\infty} \\ &= \left\| \int_u^{u'} \left(\frac{\partial P^v}{\partial v} e^{-t\Delta_{X,v}} + P^v \frac{\partial}{\partial v} (e^{-t\Delta_{X,v}}) \right) dv \right\|_{1,\infty} \\ &= \left\| \int_u^{u'} \left(\frac{\partial P^v}{\partial v} P^v e^{-t\Delta_{X,v}} + P^v \frac{\partial}{\partial v} (e^{-t\Delta_{X,v}}) \right) dv \right\|_{1,\infty} \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \int_u^{u'} \frac{\partial P^v}{\partial v} P^v e^{-t\Delta_{X,v}} dv \right\|_{1,\infty} + \left\| \int_u^{u'} P^v \left(\frac{\partial}{\partial v} (\mathcal{O}^{-\frac{t}{2}\Delta_{X,v}}) e^{-\frac{t}{2}\Delta_{X,v}} + e^{-\frac{t}{2}\Delta_{X,v}} \frac{\partial}{\partial v} (\mathcal{O}^{-\frac{t}{2}\Delta_{X,v}}) \right) dv \right\|_{1,\infty} \\
&= \left\| \int_u^{u'} \frac{\partial P^v}{\partial v} P^v e^{-t\Delta_{X,v}} dv \right\|_{1,\infty} + \left\| \int_u^{u'} \left(P^v \frac{\partial}{\partial v} (\mathcal{O}^{-\frac{t}{2}\Delta_{X,v}}) \right) (P^v e^{-\frac{t}{2}\Delta_{X,v}}) + \left(P^v e^{-\frac{t}{2}\Delta_{X,v}} \right) \frac{\partial}{\partial v} (\mathcal{O}^{-\frac{t}{2}\Delta_{X,v}}) dv \right\|_{1,\infty} \\
&\leq \int_u^{u'} \left\| \frac{\partial P^v}{\partial v} \right\|_{L^2,\infty} \theta_{v,\infty}(t) dv + 2 \int_u^{u'} \left\| \frac{\partial}{\partial v} e^{-\frac{t}{2}\Delta_{X,v}} \right\|_{L^2,\infty} \theta_{v,\infty}\left(\frac{t}{2}\right) dv \quad \text{by (4.9)} \\
&\leq c_7 \int_u^{u'} \frac{1}{2^v} \theta_{v,\infty}(t) dv + c_6 \int_u^{u'} \frac{1}{2^v} t^{\frac{1}{4}} \theta_{v,\infty}\left(\frac{t}{2}\right) dv \\
&\leq c_8 \frac{1}{2^u} \int_u^{u'} \left(\theta_{v,\infty}(t) + t^{\frac{1}{4}} \theta_{v,\infty}\left(\frac{t}{2}\right) \right) dv.
\end{aligned}$$

We used the following facts: $\frac{\partial}{\partial v}(e^{-t\Delta_{X,v}}) = \frac{\partial}{\partial v}(e^{-t\Delta_{X,v}})P^v$ and $\frac{\partial P^v}{\partial v}P^v = \frac{\partial P^v}{\partial v}Id$. Let us prove the first one: Let $v > 1$, and $\xi \in \mathcal{H}$. There exists $a(\xi)$, a constant, such that $\xi - P^v\xi = a(\xi)$, then:

$$\frac{e^{-t\Delta_{X,u}} - e^{-t\Delta_{X,u'}}}{u - u'} (\xi - P^v\xi) = \frac{1}{u - u'} (a(\xi) - a(\xi)) = 0 \quad \forall u \neq u',$$

(since $e^{-t\Delta_{X,u}}\phi = e^{-t\lambda}\phi$, if $\Delta_{X,u}\phi = \lambda\phi$). Therefore, for any $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$:

$$\left| \int_0^\infty t^{s-1} \theta_{u,\infty}(t) dt - \int_0^\infty t^{s-1} \theta_{u',\infty}(t) dt \right| \leq c_8 \int_u^{u'} \frac{1}{2^v} \int_0^\infty \left(t^{\text{Re}(s)-1} \theta_{v,\infty}(t) + t^{\text{Re}(s)+\frac{1}{4}-1} \theta_{v,\infty}\left(\frac{t}{2}\right) \right) dt dv.$$

Now, we consider the following function $\zeta_{u,\infty}$:

$$\zeta_{u,\infty}(s) := \frac{1}{\Gamma(s)} \int_0^\infty \theta_{u,\infty}(t) t^{s-1} dt, \quad \forall s \in \mathbb{C}.$$

If we let $B = P^u$ in (28), we get

$$\frac{1-\varepsilon}{1+\varepsilon} \theta_{u,\infty}(t) \leq \theta_{X,u}(t) \leq \frac{1+\varepsilon}{1-\varepsilon} \theta_{u,\infty}(t), \quad \forall t > 0, \quad (17)$$

this yields to

$$\frac{1-\varepsilon}{1+\varepsilon} \zeta_{u,\infty}(s) \leq \zeta_{X,u}(s) \leq \frac{1+\varepsilon}{1-\varepsilon} \zeta_{u,\infty}(s), \quad \forall s \in \mathbb{R}. \quad (18)$$

Since $\zeta_{X,u}(s)$ is finite for any $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$ and any $u \geq 1$, it follows that $\zeta_{\infty,u}$ is finite for any $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$. Then, for any $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, we have

$$\left| \zeta_{u,\infty}(s) - \zeta_{u',\infty}(s) \right| \leq \frac{c_8}{2^u} \int_u^{u'} \left(\frac{\Gamma(\text{Re}(s))}{|\Gamma(s)|} \zeta_{v,\infty}(\text{Re}(s)) + 2^{\text{Re}(s)+\frac{1}{4}} \frac{\Gamma(\text{Re}(s)+\frac{1}{4})}{|\Gamma(s)|} \zeta_{v,\infty}(\text{Re}(s)+\frac{1}{4}) \right) dv$$

We claim that, it is possible to suppose that the first nonzero eigenvalue of $\Delta_{X,v}$ is > 1 , for any $v > 1$. Indeed, let $v > 1$. If we multiply $\omega_{X,v}$ by $t > 0$, then the corresponding first nonzero eigenvalue is $\frac{1}{t}\lambda_{v,1}$. And, from (5.3) and (35), we have $\frac{1}{4}(\frac{c_1}{c_2})^2 h_{tg_1}(X)^2 \leq \frac{1}{4}h_{g_v}(X)^2 \leq \frac{1}{t}\lambda_{v,1}$ for any $v \geq 1$. Since, $h_{tg_1}(X) = t^{-\frac{1}{2}}h_{g_1}(X)$, we obtain: $\frac{1}{4t}(\frac{c_1}{c_2})^2 h_{g_1}(X)^2 \leq \frac{1}{t}\lambda_{v,1}$ for any $v \geq 1$. Hence, for t sufficiently small, we can assume that $\lambda_{v,1} > 1$ for any $v \geq 1$.

Therefore, we get for any $s > 1$:

$$\begin{aligned}\zeta_{X,v}(s) - \zeta_{X,v}(s + \frac{1}{4}) &= \sum_{k=1}^{\infty} \frac{1}{\lambda_{v,k}^s} - \sum_{k=1}^{\infty} \frac{1}{\lambda_{v,k}^{s+\frac{1}{4}}} \\ &\geq \sum_{k=1}^{\infty} \frac{1}{\lambda_{v,k}^s} - \frac{1}{\lambda_{v,1}^s} \sum_{k=1}^{\infty} \frac{1}{\lambda_{v,k}^s} \\ &= \left(1 - \frac{1}{\lambda_{1,v}^{\frac{1}{4}}}\right) \zeta_{X,v}(s).\end{aligned}$$

Then, using (18), it is possible to have:

$$\zeta_{v,\infty}(s) \geq \zeta_{v,\infty}(s + \frac{1}{4}), \quad \forall s > 1, \forall v \gg 1.$$

Therefore,

$$\left| \zeta_{u,\infty}(s) - \zeta_{u',\infty}(s) \right| \leq \frac{c_8}{2^u} \left(\frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} + 2^{\operatorname{Re}(s)+\frac{1}{4}} \frac{\Gamma(\operatorname{Re}(s) + \frac{1}{4})}{|\Gamma(s)|} \right) \int_u^{u'} \zeta_{v,\infty}(\operatorname{Re}(s)) dv, \quad \forall \operatorname{Re}(s) > 1. \quad (19)$$

Now we assume $s > 1$. Let us prove that, for any $\forall u, u' \gg 1$ and any $s > 1$:

$$\exp\left(-\frac{c_8}{\log 2} \left| \frac{1}{2^u} - \frac{1}{2^{u'}} \right| \left(\frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} + 2^{s+\frac{1}{4}} \frac{\Gamma(s+\frac{1}{4})}{\Gamma(s)} \right) \right) \leq \frac{\zeta_{u,\infty}(s)}{\zeta_{u',\infty}(s)} \leq \exp\left(\frac{c_8}{\log 2} \left| \frac{1}{2^u} - \frac{1}{2^{u'}} \right| \left(\frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} + 2^{s+\frac{1}{4}} \frac{\Gamma(s+\frac{1}{4})}{\Gamma(s)} \right) \right). \quad (20)$$

If $u' \mapsto \zeta_{u',\infty}(s)$ is a derivable function for s fixed, we obtain using (19)

$$\left| \frac{\partial}{\partial u} \zeta_{u,\infty}(s) \right| \leq \frac{c_8}{2^u} \left(\frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} + 2^{\operatorname{Re}(s)+\frac{1}{4}} \frac{\Gamma(\operatorname{Re}(s) + \frac{1}{4})}{|\Gamma(s)|} \right) \zeta_{u,\infty}(s),$$

then,

$$\left| \frac{\partial}{\partial u} \log \zeta_{u,\infty}(s) \right| \leq \frac{c_8}{2^u} \left(\frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} + 2^{s+\frac{1}{4}} \frac{\Gamma(s+\frac{1}{4})}{\Gamma(s)} \right),$$

which gives:

$$\left| \log \zeta_{u,\infty}(s) - \log \zeta_{u',\infty}(s) \right| \leq \frac{c_8}{\log 2} \left| \frac{1}{2^u} - \frac{1}{2^{u'}} \right| \left(\frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} + 2^{s+\frac{1}{4}} \frac{\Gamma(s+\frac{1}{4})}{\Gamma(s)} \right) \quad \forall u, u'.$$

On the other hand, if $u' \mapsto \zeta_{u',\infty}(s)$ is not derivable, we apply the Gronwall lemma to the function $u' \mapsto \zeta_{u',\infty}(s)$. Note that this function is continuous, because it is locally Lipschitz, which is a consequence of the continuity of $v \mapsto \zeta_{X,v}(s)$, and the inequalities (18) and (19).

From (18) and (20), there exist positives constants c_{12} and c_{13} such that

$$c_{12} \exp\left(-\frac{c_8}{\log 2} \left| \frac{1}{2^u} - \frac{1}{2^{u'}} \right| \left(\frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} + 2^{s+\frac{1}{4}} \frac{\Gamma(s+\frac{1}{4})}{\Gamma(s)} \right) \right) \leq \frac{\zeta_{X,u}(s)}{\zeta_{X,u'}(s)} \leq c_{13} \exp\left(\frac{c_8}{\log 2} \left| \frac{1}{2^u} - \frac{1}{2^{u'}} \right| \left(\frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} + 2^{s+\frac{1}{4}} \frac{\Gamma(s+\frac{1}{4})}{\Gamma(s)} \right) \right) \quad (21)$$

for any $u, u' \gg 1$ et $\forall s > 1$.

We know that $\zeta_{X,u}$ is holomorphic on the open set $\{s \in \mathbb{C} | \operatorname{Re}(s) > 1\}$. Let us prove that $(\zeta_{X,u})_{u \geq 1}$ converges to a holomorphic function defined on $\{s \in \mathbb{C} | \operatorname{Re}(s) > 1\}$ uniformly on any domain of the form

$\alpha \leq \operatorname{Re}(s) \leq \beta$, where $1 < \alpha \leq \beta$. We will show that this function is in fact $\zeta_{X,\infty}$.

From (17), we have

$$-\frac{2\varepsilon}{1+\varepsilon}\theta_{\infty,u}(t) \leq \theta_{X,u}(t) - \theta_{\infty,u}(t) \leq \frac{2\varepsilon}{1-\varepsilon}\theta_{u,\infty} \quad \forall t > 0 \quad \forall u, u' \gg 1$$

Then,

$$\left| \theta_{X,u}(t) - \theta_{\infty,u}(t) \right| \leq \frac{2\varepsilon}{1-\varepsilon}\theta_{u,\infty}(t),$$

It follows that,

$$\left| \zeta_{X,u}(s) - \zeta_{u,\infty}(s) \right| \leq \frac{2\varepsilon}{1-\varepsilon} \frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} \zeta_{u,\infty}(\operatorname{Re}(s)) \quad \forall \operatorname{Re}(s) > 1.$$

Using this inequality, we get:

$$\left| \zeta_{X,u}(s) - \zeta_{X,u'}(s) \right| \leq \left| \zeta_{u,\infty}(s) - \zeta_{u',\infty}(s) \right| + \frac{2\varepsilon}{1-\varepsilon} \frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} \left(\zeta_{u,\infty}(\operatorname{Re}(s)) + \zeta_{u',\infty}(\operatorname{Re}(s)) \right),$$

for any $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > 1$ and for any $u, u' \gg 1$.

If we choose $1 < \alpha < \beta$. Then, we have already proved that $(\zeta_{u,\infty}(\operatorname{Re}(s)))_u$ is uniformly bounded with respect to u on $\alpha \leq \operatorname{Re}(s) \leq \beta$. Then, we can find a constant K , which depend uniquely on α and β such that

$$\left| \zeta_{X,u}(s) - \zeta_{X,u'}(s) \right| \leq \left| \zeta_{u,\infty}(s) - \zeta_{u',\infty}(s) \right| + \frac{2\varepsilon}{1-\varepsilon} K,$$

for any $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > 1$ and for any $u, u' \gg 1$.

Therefore, $(\zeta_{X,u})_{u \geq 1}$ converges uniformly to a limit on the set $\{s \in \mathbb{C} \mid \alpha \leq \operatorname{Re}(s) \leq \beta\}$, and this limit is necessarily holomorphic on $\{s \in \mathbb{C} \mid \alpha \leq \operatorname{Re}(s) \leq \beta\}$.

The following claim will be used to establish that $(\zeta_{X,u})_{u \geq 1}$ converges pointwise to $\zeta_{X,\infty}$ on the set $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1\}$.

Claim 3.20. *Let θ be a positive decreasing function on \mathbb{R}^+ . We let ζ be the function given by:*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \theta(t) dt,$$

for $s \in \mathbb{R}$. We have,

$$\theta(a) \leq \frac{\Gamma(s+1)}{a^s} \zeta(s), \quad \forall s > a > 0. \quad (22)$$

Proof. Let $a > 0$ and $s > a$, we have

$$\begin{aligned} \zeta(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \theta(t) t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^a \theta(t) t^{s-1} dt + \frac{1}{\Gamma(s)} \int_a^\infty \theta(t) t^{s-1} dt \\ &\geq \frac{\theta(a)}{\Gamma(s)} \int_0^a t^{s-1} dt \\ &= \frac{\theta(a)}{\Gamma(s+1)} a^s. \end{aligned}$$

□

We will apply this lemma in order to show that $\theta_{X,u}(t)$ uniformly bounded with respect to u for any $t > 0$. From (22), we have,

$$\theta_{X,u}(t) \leq \frac{\Gamma(s+1)}{t^s} \zeta_{X,u}(s), \quad \forall s > t > 0.$$

then, if we choose $s > 1$ so $\zeta_{X,u}(s)$ is finite. Using (21), we deduce that for any fixed $t > 0$, $\theta_{X,u}(t)$ is uniformly bounded with respect to u .

From (28), (29) and (30) we get

$$(\theta_{X,u}(t))_u \xrightarrow{u \rightarrow \infty} \theta_{X,\infty}(t) \quad \forall t > 0.$$

And using (3.19), we deduce easily that:

$$(\rho_{X,u}(t))_u \xrightarrow{u \rightarrow \infty} \rho_{X,\infty}(t) \quad \forall t > 0.$$

Fix $\varepsilon > 0$, since $(\zeta_{X,u}(1+\varepsilon))_{u \geq 1}$ is a convergent sequence, we can find a constant c such that:

$$\theta_{X,u}(t) \leq \frac{1}{t^{\varepsilon+1}} c, \quad \forall 0 < t < 1 + \varepsilon, \forall u \gg 1.$$

then,

$$\theta_{X,\infty}(t) \leq \frac{1}{t^{\varepsilon+1}} c.$$

Let $s \in \mathbb{C}$ such that $\text{Re}(s) > 1 + \varepsilon$. We have

$$\zeta_{X,u}(s) = \frac{1}{\Gamma(s)} \int_0^\delta t^{s-\varepsilon-2} (t^{\varepsilon+1} \theta_{X,u}(t)) dt + \frac{1}{\Gamma(s)} \int_\delta^\infty t^{s-1} \theta_{X,u}(t) dt, \quad \forall u \geq 1.$$

Since $(\theta_{X,u})_u$ converges pointwise to θ_∞ , and by the dominated convergence theorem, we get:

$$(\zeta_{X,u}(s))_{u \geq 1} \xrightarrow{u \rightarrow \infty} \zeta_{X,\infty}(s) \quad \forall s \in \mathbb{C} \text{ s.t. } \text{Re}(s) > 1 + \varepsilon.$$

Let us prove the following

$$\zeta_{X,\infty}(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_{\infty,k}^s},$$

for any $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$. Let us show first

$$\zeta_{X,\infty}(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_{\infty,k}^s}, \quad \forall s > 1.$$

Let $\delta > 0$, we have for any $s > 1 + \varepsilon$:

$$\begin{aligned} \zeta_{X,\infty}(s) &= \frac{1}{\Gamma(s)} \int_0^\delta t^{s-1} \theta_{X,\infty}(t) dt + \int_\delta^\infty t^{s-1} \theta_{X,\infty}(t) dt \\ &= \frac{1}{\Gamma(s)} \int_0^\delta (\theta_{X,\infty}(t) t^{\varepsilon+1}) t^{s-2-\varepsilon} dt + \int_\delta^\infty t^{s-1} \theta_{X,\infty}(t) dt. \end{aligned}$$

Since $\left| \frac{1}{\Gamma(s)} \int_0^\delta \left(\theta_{X,\infty}(t) t^{\varepsilon+1} \right) t^{s-2-\varepsilon} dt \right| \leq \frac{c}{s-1-\varepsilon} \delta^{s-1-\varepsilon}$ and $\theta_{X,\infty}(t) \leq \theta_{X,\infty}(\delta) e^{-\lambda_{\infty,1}(t-\delta)}$ for any $t \geq \delta$, then

$$\begin{aligned} \zeta_{X,\infty}(s) &= O(\delta^{s-1-\varepsilon}) + \sum_{k=1}^{\infty} \frac{1}{\Gamma(s)} \int_{\delta}^{\infty} t^{s-1} e^{-\lambda_{\infty,k} t} dt \\ &\leq O(\delta^{s-1-\varepsilon}) + \sum_{k=1}^{\infty} \frac{1}{\lambda_{\infty,k}^s} \frac{1}{\Gamma(s)} \int_{\lambda_{\infty,k} \delta}^{\infty} t^{s-1} e^{-t} dt \\ &\leq O(\delta^{s-1-\varepsilon}) + \sum_{k=1}^{\infty} \frac{1}{\lambda_{\infty,k}^s}. \end{aligned}$$

Remark that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_{\infty,k}^s} = \lim_{N \rightarrow \infty} \sum_{k=1}^N \frac{1}{\lambda_{\infty,k}^s} \leq \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \theta_{X,\infty}(s) dt = \zeta_{X,\infty}(s).$$

We conclude that

$$\zeta_{X,\infty}(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_{\infty,k}^s}, \quad \forall s > 1.$$

Let $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > 1$. For any $N \in \mathbb{N}$, we set

$$\zeta_{N,\infty}(s) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{k=N}^{\infty} e^{-\lambda_{\infty,k} t} \right) dt,$$

then

$$\zeta_{N,\infty}(s) = \zeta_{X,\infty}(s) - \sum_{k=1}^{N-1} \frac{1}{\lambda_{\infty,k}^s}, \quad \forall \operatorname{Re}(s) > 1.$$

We have,

$$|\zeta_{N,\infty}(s)| \leq \left| \frac{1}{\Gamma(s)} \right| \int_0^{\infty} t^{\operatorname{Re}(s)-1} \left(\sum_{k=N}^{\infty} e^{-\lambda_{\infty,k} t} \right) dt = \frac{\Gamma(\operatorname{Re}(s))}{|\Gamma(s)|} \left(\zeta_{X,\infty}(\operatorname{Re}(s)) - \sum_{k=1}^{N-1} \frac{1}{\lambda_{\infty,k}^{\operatorname{Re}(s)}} \right).$$

The right-hand side converges to zero when N goes to infinity. We conclude that

$$\zeta_{X,\infty}(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_{\infty,k}^s}, \quad \forall s \in \mathbb{C} \text{ s.t. } \operatorname{Re}(s) > 1.$$

Let $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > 1$, we have

$$\begin{aligned} |\zeta_{X,u}(s) - \zeta_{X,u'}(s)| &\leq |\zeta_{X,u}(s)| + |\zeta_{X,u'}(s)| \\ &\leq \zeta_{X,u}(\operatorname{Re}(s)) + \zeta_{X,u'}(\operatorname{Re}(s)). \end{aligned}$$

(because $\zeta_{X,u}(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_{u,k}^s}$ when $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > 1$).

Let $x \in \mathbb{C}$ such that $\operatorname{Re}(x) > 0$. For any $u \geq 1$, we set:

$$\tilde{\theta}_u(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \Gamma(s) \zeta_{X,u}(s) ds,$$

where c a fixed integer greater than 1.

One verify the following assertions: $\tilde{\theta}_u(x) = \sum_{k \geq 1} e^{-\lambda_{u,k}x}$, $|\tilde{\theta}_u(x)| \leq \theta(\operatorname{Re}(x))$, $\tilde{\theta}_u$ and $\theta_{X,u}$ are equal on \mathbb{R}^{+*} , and that $\tilde{\theta}_u(x) = \frac{a-1}{x} + a_0 + \tilde{\rho}(x)$ for x small enough.

We have,

$$\begin{aligned}
\left| \tilde{\theta}_u(x) - \tilde{\theta}_{u'}(x) \right| &\leq \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left| \left(x^{-s} \Gamma(s) \zeta_{X,u}(s) - x^{-s} \Gamma(s) \zeta_{X,u'}(s) \right) \right| ds \\
&\leq \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} |x|^{\operatorname{Re}(s)} |\Gamma(s)| \left(\zeta_{X,u}(\operatorname{Re}(s)) + \zeta_{X,u'}(\operatorname{Re}(s)) \right) ds \\
&= \frac{1}{2\pi} |x|^{-c} \left(\zeta_{X,u}(c) + \zeta_{X,u'}(c) \right) \left(\int_{-1}^1 |\Gamma(c+it)| dt + \int_{]-\infty, -1] \cup [1, \infty[} |\Gamma(c+it)| dt \right) \\
&= \frac{1}{2\pi} |x|^{-c} \left(\zeta_{X,u}(c) + \zeta_{X,u'}(c) \right) \left(\int_{-1}^1 |\Gamma(c+it)| dt + \int_{]-\infty, -1] \cup [1, \infty[} \prod_{k=0}^{c-1} \sqrt{k^2 + t^2} |\Gamma(it)| dt \right) \\
&= \frac{1}{2\pi} |x|^{-c} \left(\zeta_{X,u}(c) + \zeta_{X,u'}(c) \right) \left(\int_{-1}^1 |\Gamma(c+it)| dt + \int_{]-\infty, -1] \cup [1, \infty[} \prod_{k=0}^{c-1} \sqrt{k^2 + t^2} \frac{\sqrt{\pi}}{\sqrt{|t|} \sqrt{|\sinh(\pi t)|}} |dt \right) \\
&\quad \text{see formula [1, 6.1.29].}
\end{aligned}$$

We verify that the last integral is convergent. Since $(\zeta_{X,u})_{u \geq 1}$ converges uniformly on any domain of the form $\{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq \delta + 1\}$. Then, there exists constant K which depend only on c , such that:

$$|\tilde{\theta}_u(x) - \tilde{\theta}_{u'}(x)| \leq K|x|^{-c}, \quad \forall x \in \mathbb{C} \text{ s.t. } \operatorname{Re}(x) > 0.$$

We conclude that

$$|\tilde{\rho}_u(x) - \tilde{\rho}_{u'}(x)| \leq K|x|^{-c} + |a_{u,-1} - a_{u',-1}| |x|^{-1}, \quad \forall x \in \mathbb{C} \text{ s.t. } \operatorname{Re}(x) > 0.$$

Let $r > 0$ fixed. We denote by D the curve in the complex plan, given by $re^{i\alpha}$, such that $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$.

If we replace x by x^2 in the above inequality, and we consider k , an integer greater than 1, then we get

$$\left| \int_D \frac{\tilde{\rho}_u(x^2) - \tilde{\rho}_{u'}(x^2)}{x^{2k}} dx \right| \leq K \int_D |x|^{-2c-2k} dx + |a_{u,-1} - a_{u',-1}| \int_D |x|^{-2-2k} dx,$$

this yields to

$$|a_{u,k} - a_{u',k}| \leq K r^{-2c-2k} + |a_{u,-1} - a_{u',-1}| r^{-2-2k} \quad \forall u, u' \forall k \in \mathbb{N}_{\geq 1},$$

(we have used the following fact: $\int_D x^{2(j-k)} dx = \delta_{k,j} \pi$, and recall that $a_{u,0}$ does not depend on u).

Therefore, if we take $0 < t < r$, we get

$$\begin{aligned}
|\rho_{X,u}(t^2)| &\leq \sum_{k \geq 1} |a_{u,k}| t^{2k} \\
&\leq \sum_{k \geq 1} |a_{u,k} - a_{u',k}| t^{2k} + \sum_{k=1}^{\infty} |a_{u',k}| t^{2k}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k \geq 1} K r^{-2c} \left(\frac{t^2}{r^2} \right)^k + \sum_{k \geq 1} r^{-2} \left(\frac{t^2}{r^2} \right)^k |a_{u,-1} - a_{u',-1}| + \sum_{k=1}^{\infty} |a_{u',k}| t^{2k} \\
&\leq K r^{-2c} \frac{t^2}{r^2 - t^2} + r^{-2} \frac{t^2}{r^2 - t^2} |a_{u,-1} - a_{u',-1}| + \sum_{k=1}^{\infty} |a_{u',k}| t^{2k}.
\end{aligned}$$

Now, recall that $(a_{u,-1})_{u \geq 1}$ is bounded, see (3.19). If we fix u' , we can find a constant K' and a real $t_0 > 0$ such that

$$\sum_{k=1}^{\infty} |a_{u',k}| t^{2k} \leq K' \frac{t^2}{1 - t^2} \quad \forall t \in [0, t_0].$$

We conclude there exists a constant K'' such that:

$$|\rho_{X,u}(t)| \leq K'' t, \quad \forall u \gg 1, \forall 0 \leq t \leq \min(\sqrt{r}, t_0).$$

But, we proved that ρ_u converges pointwise to ρ_{∞} , then

$$|\rho_{X,\infty}(t)| \leq K'' t, \quad \forall 0 \leq t \leq \min(\sqrt{r}, t_0).$$

that is $\rho_{X,\infty}(t) = O(t)$. As a first consequence, $\zeta_{X,\infty}$ admits a holomorphic continuation at $s = 0$, and

$$\zeta'_{X,\infty}(0) = \int_1^{\infty} \frac{\theta_{X,\infty}(t)}{t} dt + a_{\infty,-1} + a_{\infty,0} + \int_0^1 \frac{\rho_{X,\infty}(t)}{t} dt.$$

Since,

$$\theta_{X,u}(t) \leq \theta_{X,u}(1) e^{-\lambda_{u,1}(t-1)} \quad \forall t \geq 1.$$

and from (5.6), we can find a constant $\kappa > 0$ such that $\lambda_{u,1} \geq \kappa$ for any $u \geq 1$. Recall that $(\theta_{X,u}(1))_u$ is bounded. Therefore, there exists $M > 0$ such that

$$\theta_{X,u}(t) \leq M e^{-\kappa t} \quad \forall t \geq 1.$$

Also, we proved that

$$|\rho_{X,u}(t)| \leq K'' t, \quad \forall u \gg 1 \forall 0 \leq t \leq \min(\sqrt{r}, t_0).$$

We deduce, using the dominated convergence theorem, that

$$\left(\int_1^{\infty} \frac{\theta_{X,u}(t)}{t} dt \right)_u \xrightarrow{u \rightarrow \infty} \int_1^{\infty} \frac{\theta_{X,\infty}(t)}{t} dt,$$

and

$$\left(\int_0^1 \frac{\rho_{X,u}(t)}{t} dt \right)_u \xrightarrow{u \rightarrow \infty} \int_0^1 \frac{\rho_{X,\infty}(t)}{t} dt.$$

Then,

$$(\zeta'_{X,u}(0))_{u \geq 1} \xrightarrow{u \rightarrow \infty} \zeta'_{X,\infty}(0).$$

In particular,

$$(\zeta'_{X,p}(0))_{p \in \mathbb{N}} \xrightarrow{p \rightarrow \infty} \zeta'_{X,\infty}(0). \tag{23}$$

□

Theorem 3.21. *We keep the same assumptions. For any $p \in \mathbb{N}$, let $h_{Q,((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))}$ be the Quillen metric associated to $((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))$. We have, the sequence $\left(h_{Q,((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))}\right)_{p \in \mathbb{N}}$ converges to a limit, which does not depend on the choice of $(h_{X,p})_{p \in \mathbb{N}}$. We denote this limit by $h_{Q,((X,\omega_{X,\infty});(\mathcal{O},h_{\mathcal{O}}))}$.*

Proof. By the anomaly formula, see [3], we have for any $p, q \in \mathbb{N}$:

$$-\log h_{Q,((TX,th_{X,p});(\mathcal{O},h_{\mathcal{O}}))} + \log h_{Q,((TX,h_{X,q});(\mathcal{O},h_{\mathcal{O}}))} = \int_X ch(\mathcal{O}, h_{\mathcal{O}}) \widetilde{\text{Td}}(TX, h_{X,p}, h_{X,q}).$$

So, it suffices to prove that the right hand side forms a Cauchy sequence. We have:

$$\widetilde{\text{Td}}(TX, h_{X,p}, h_{X,q}) = -\frac{1}{2} \log \left(\frac{h_{X,p}}{h_{X,q}} \right) - \frac{1}{12} \log \left(\frac{h_{X,p}}{h_{X,q}} \right) \left(c_1(TX, h_{X,p}) + c_1(TX, h_{X,q}) \right).$$

in $\oplus_j \tilde{A}^{j,j}(X)$, that is the algebra of $(*,*)$ -differential forms on X , modulo $\text{Im } \bar{\partial} + \text{Im } \partial$, see [8].

Then,

$$\int_X ch(\mathcal{O}, h_{\mathcal{O}}) \widetilde{\text{Td}}(TX, h_{X,p}, h_{X,q}) = -\frac{1}{12} \int_X \log \left(\frac{h_{X,p}}{h_{X,q}} \right) \left(c_1(TX, h_{X,p}) + c_1(TX, h_{X,q}) \right)$$

Since, for any $n \in \mathbb{N}$, there exist $(h_{n,1})_{n \in \mathbb{N}}$ and $(h_{n,2})_{n \in \mathbb{N}}$ two smooth positive metrics such that $h_{X,n} = h_{n,1} \otimes h_{n,2}^{-1}$ for any $n \in \mathbb{N}$. Then, we can find a constant K , such that:

$$\left| \int_X ch(\mathcal{O}, h_{\mathcal{O}}) \widetilde{\text{Td}}(TX, h_{X,p}, h_{X,q}) \right| \leq K \left\| \log \left(\frac{h_{X,p}}{h_{X,q}} \right) \right\|_{\text{sup}}.$$

It follows that $\left(h_{Q,((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))}\right)_{p \in \mathbb{N}}$ admits a finite limit. From the previous inequality, it is clear that the limit does not depend on the choice of $(h_{X,p})_{p \in \mathbb{N}}$. \square

The last theorem extend the notion of Quillen metrics to integrable metrics on compact riemannian surfaces, by using an approximation process. However, theorem (3.17) provides a direct way to define the holomorphic analytic torsion in this situation. It is important to compare both methods, this is given in the following theorem:

Theorem 3.22. *We have,*

$$h_{Q,((X,\omega_{X,\infty});(\mathcal{O},h_{\mathcal{O}}))} = h_{L^2,((X,\omega_{X,\infty});(\mathcal{O},h_{\mathcal{O}}))} \exp(\zeta'_{X,\infty}(0)).$$

Proof. We have, for any $p \in \mathbb{N}_{\geq 1}$:

$$h_{Q,((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))} = h_{L^2,((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))} \exp(\zeta'_{X,p}(0)).$$

According to (23), it suffices to prove that $\left(h_{L^2,((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))}\right)_{p \in \mathbb{N}}$ converges to $h_{L^2,((X,\omega_{X,\infty});(\mathcal{O},h_{\mathcal{O}}))}$. Indeed, we have $h_{L^2,((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))} = \text{Vol}_{L^2,p}(H^0(X, \mathcal{O})) \text{Vol}_{L^2,p}(H^1(X, \mathcal{O}))^{-1}$, for any $p \in \mathbb{N}$. One verify easily the convergence of $\left(\text{Vol}_{L^2,p}(H^0(X, \mathcal{O}))\right)_{p \in \mathbb{N}}$ to $\text{Vol}_{L^2,\infty}(H^0(X, \mathcal{O}))$. The convergence of $\left(\text{Vol}_{L^2,p}(H^1(X, \mathcal{O}))\right)_{p \in \mathbb{N}}$ follows from (3.24). \square

We need the following lemma, in order to prove (3.24):

Lemma 3.23. Let (X, ω_X) be a compact riemannian surface. Let $\overline{\mathcal{O}} = (\mathcal{O}, h_{\mathcal{O}})$ a holomorphic line bundle endowed with a constant hermitian metric $h_{\mathcal{O}}$. We let $K_X = \Omega_X^{(1,0)}$ which we equip with the induced metric from ω_X . We have:

$$\Delta_{K_X}^0 *_1 = *_1 \Delta_{\mathcal{O}}^1.$$

where Δ_{\bullet}^* is the Laplacian acting on $A^{(0,*)}(X, \bullet)$. In particular,

$$\ker(\Delta_{\mathcal{O}}^1) = *_1^{-1} \left(H^0(X, K_X) \right).$$

Proof. Let $\xi \in A^{(0,1)}(X)$. We have (see paragraph (2) for the notations):

$$\begin{aligned} \Delta_{K_X}(*_1 \xi) &= \overline{\partial}_{K_X}^* \overline{\partial}_{K_X}(*_1 \xi) \\ &= *_1 \overline{\partial}_{\mathcal{O}} *_0^{-1} \overline{\partial}_{K_X} *_1 \xi \\ &= *_1 \overline{\partial}_{\mathcal{O}} \overline{\partial}_{\mathcal{O}}^* \xi \\ &= *_1 \Delta_{\mathcal{O}}^1 \xi. \end{aligned}$$

Since $\ker \Delta_{K_X}^0 = H^0(X, K_X)$, then

$$\ker \Delta_{\mathcal{O}}^1 = *_1^{-1} \left(H^0(X, K_X) \right).$$

□

Lemma 3.24. Let X compact riemannian surface. Let $(h_p)_{p \in \mathbb{N}}$ be a sequence of continuous hermitian metrics which converges uniformly to $h_{X,\infty}$ on TX .

We have

$$\left(\text{Vol}_{L^2,p}(H^1(X, \mathcal{O})) \right)_{p \in \mathbb{N}} \xrightarrow{p \rightarrow \infty} \text{Vol}_{L^2,\infty}(H^1(X, \mathcal{O})).$$

Proof. From (1), we see that $*_1$ does not depend on the metric of X . Let $\xi, \xi' \in A^{(0,1)}(X, K_X)$. Since X is compact, we may assume that $\xi = g dz$ and $\xi' = f dz$, where $g, f \in A^{(0,0)}(X)$ and z is a local coordinate. We have for any $p, q \in \mathbb{N}$

$$*_{1,p}^{-1}(g dz) = *_{1,q}^{-1}(g dz),$$

and,

$$(*_{1,p}^{-1} \xi, *_{1,p}^{-1} \xi')_{L_{X,q}^2} = \frac{i}{2\pi} \int_X \overline{g} f dz \wedge d\overline{z}.$$

Therefore, obviously we have

$$\left((*_{1,p}^{-1} \xi, *_{1,p}^{-1} \xi')_{L_{X,p}^2} \right)_{p \in \mathbb{N}} \xrightarrow{p \rightarrow \infty} (*_{1,\infty}^{-1} \xi, *_{1,\infty}^{-1} \xi')_{L_{X,\infty}^2}.$$

By proposition (3.23), we have

$$\ker \Delta_{X,p}^1 = *_1^{-1} \left(H^0(X, K_X) \right) \quad \forall p \in \mathbb{N}_{\geq 1}.$$

Then,

$$\left(\text{Vol}_{L^2,p}(H^1(X, \mathcal{O})) \right)_{p \in \mathbb{N}} \xrightarrow{p \rightarrow \infty} \text{Vol}_{L^2,\infty}(H^1(X, \mathcal{O})).$$

□

We conclude that:

$$\left(h_{L^2,((X,\omega_{X,p});(\mathcal{O},h_{\mathcal{O}}))} \right)_{p \in \mathbb{N}} \xrightarrow{p \rightarrow \infty} h_{L^2,((X,\omega_{X,\infty});(\mathcal{O},h_{\mathcal{O}}))}.$$

4 Appendix 1

4.1 Integrable metrics

Let X be a complex analytic manifold and $\overline{L} = (L, \|\cdot\|)$ holomorphic line bundle equipped with a continuous hermitian metric on L .

Definition 4.1. We call first Chern current of \overline{L} , and we denote it by $c_1(\overline{L}) \in D^{(1,1)}(X)$, the $(1,1)$ -current defined locally by the following formula:

$$c_1(\overline{L}) = c_1(L, \|\cdot\|) = dd^c(-\log \|s\|^2),$$

where s is a nonzero holomorphic local section of L .

Definition 4.2. The metric $\|\cdot\|$ is positive if $c_1(L, \|\cdot\|) \geq 0$.

Definition 4.3. We say $\|\cdot\|$ is admissible if there exists a sequence $(\|\cdot\|_n)_{n \in \mathbb{N}}$ of smooth positive hermitian metrics converging uniformly to $\|\cdot\|$ on L . An admissible line bundle on X is a holomorphic line bundle equipped with an admissible metric.

We say \overline{L} is an integrable line bundle if there exist \overline{L}_1 et \overline{L}_2 two admissible line bundles such that:

$$\overline{L} = \overline{L}_1 \otimes \overline{L}_2^{-1}.$$

4.2 Compacts operators

Consider two Hilbert spaces \mathcal{H} and \mathcal{H}' . We denote by $L(\mathcal{H}, \mathcal{H}')$ the space of continuous linear maps from \mathcal{H} to \mathcal{H}' .

We say that T in $L(\mathcal{H}, \mathcal{H}')$ is a compact operator if and only if the image of every bounded subset of \mathcal{H} by T is relatively compact in \mathcal{H}' . An operator T is said to be finite rank operator if its image has a finite dimension. In particular, it is a compact operator. The dimension of its image is called the rank of the operator. We denote by \langle, \rangle the inner product of \mathcal{H} , and by $\|\cdot\|$ the associated norm.

Proposition 4.4. Let \mathcal{H} be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators and $\mathcal{K}(\mathcal{H})$ the space of compacts operators. We have, $\mathcal{K}(\mathcal{H})$ is a closed linear subspace of $\mathcal{B}(\mathcal{H})$.

Proof. See for example [13, proposition 1.4]. □

Let $T \in L(\mathcal{H})$. For any $n \in \mathbb{N}$, we set

$$\sigma_n(T) = \inf \left\{ \|T - R\| : R \in L(\mathcal{H}), \text{rg}(R) \leq n \right\}, \quad (24)$$

According to [13, p. 232], T is compact if and only if the sequence $(\sigma_n(T))_{n \in \mathbb{N}}$ converges to 0. We assume that T is compact, and $\sigma_n(T)$ is called the n -th singular value of T .

Let $P := (T^*T)^{\frac{1}{2}}$ (where T^* denote adjoint operator of T), one shows that P is an positive selfadjoint compact operator; we denote by $(\mu_n(T))_{n \in \mathbb{N}}$ the set of nonzero eigenvalues P , in decreasing order and counted with multiplicity (that is, each nonzero eigenvalue λ appears d_λ times, where d_λ is the dimension of $\ker(\lambda I - P)$).

We know that

$$\mu_n(T) = \sigma_n(T) \quad \forall n \in \mathbb{N},$$

see for example [13, p. 246].

Definition 4.5. Let T be a compact operator. Let $(\mu_n(T))_{n \in \mathbb{N}}$ the set of singular values of T , in decreasing order. We set

$$\|T\|_1 := \sum_{n \in \mathbb{N}} \mu_n(T).$$

If $\|T\|_1 < \infty$ then T is called trace-class (or nuclear) operator, and $\|T\|_1$ denote its nuclear norm.

We denote by $\mathcal{C}_1(\mathcal{H})$ the set of class trace operators. We have:

Proposition 4.6. $\mathcal{C}_1(\mathcal{H})$ is vectorial space, and $\|\cdot\|_1$ is a norm on $\mathcal{C}_1(\mathcal{H})$, called the trace norm.

Proof. See [6, 15.11 problem 7, c]. □

Proposition 4.7. Let T be a trace-class operator. Let $(\xi_n)_{n \in \mathbb{N}}$ an orthonormal basis for \mathcal{H} . Then $\sum_{n \in \mathbb{N}} \langle T\xi_n, \xi_n \rangle$ converges, with sum equal to $\|T\|_1$.

Proof. See [6, 15.11 problem 7, b)]. □

Proposition 4.8. Let T be a trace-class operator, and be $(\lambda_n)_{n \in \mathbb{N}}$ the sequence of its eigenvalues counted with their multiplicity. Then, $\sum_{n \in \mathbb{N}} \lambda_n$ converges absolutely and we have

$$\sum_{n \in \mathbb{N}} \lambda_n = \text{Tr}(T).$$

Proof. See [18, (3.2)]. □

Proposition 4.9. Let A et B be two bounded operators and $T \in \mathcal{C}_1(\mathcal{H})$, then

$$\|ATB\|_1 \leq \|A\| \|T\|_1 \|B\|.$$

Let \mathcal{H} be a separable Hilbert space. Let A be a compact operator on \mathcal{H} and $(e_i)_i$ an orthonormal basis of \mathcal{H} . If $\sum_{i \geq 0} \langle Ae_i, e_i \rangle$ is absolutely convergent with respect to an orthonormal basis $(e_i)_i$, and hence for any orthonormal basis of \mathcal{H} , we call this sum the trace of A and we denote it by $\text{Tr}(A)$.

If T is class trace operator, then

$$\|T\|_1 = \text{Tr}((T^*T)^{\frac{1}{2}}).$$

Let A be a class trace operator, we have:

•
$$\text{Tr}(AB) = \text{Tr}(BA), \tag{25}$$

if B is bounded, cf. [15, TR. 2 p.463].

•
$$|\text{Tr}(A)| \leq \|A\|_1. \tag{26}$$

cf. [15, TR. 7 p.463].

Proposition 4.10. Let \mathcal{H} be a Hilbert space. let $(\langle \cdot, \cdot \rangle_u)_{u \in I}$ a family of hermitian metrics on \mathcal{H} uniformly equivalents. Let $u_0 \in I$, and $T \in \mathcal{C}_{1, u_0}(\mathcal{H})$ then

1. T is a trace-class operator on \mathcal{H} endowed with the norm $\langle \cdot, \cdot \rangle_u$, for any $u \in I$,
2. There exists c_8 et c_9 positives constants such that:

$$c_8 \|T\|_{1, u'} \leq \|T\|_{1, u} \leq c_9 \|T\|_{1, u'} \quad \forall u, u' \in I.$$

Proof. Let us recall that $L(\mathcal{H})$ is endowed with the following norm:

$$\|A\| = \sup_{x \in \mathcal{H} \setminus \{0\}} \frac{\|Ax\|}{\|x\|} \quad A \in L(\mathcal{H}).$$

By assumption, there exist c'_8, c'_9 two positive constants such that:

$$c'_8 \|x\|_{u'} \leq \|x\|_u \leq c'_9 \|x\|_{u'} \quad \forall x \in \mathcal{H} \quad \forall u, u' \in I.$$

Hence,

$$\frac{c'_8}{c'_9} \frac{\|Tx\|_{u'}}{\|x\|_{u'}} \leq \frac{\|Tx\|_u}{\|x\|_u} \leq \frac{c'_9}{c'_8} \frac{\|Tx\|_{u'}}{\|x\|_{u'}} \quad \forall x \neq 0.$$

Therefore,

$$\frac{c'_8}{c'_9} \|T\|_{u'} \leq \|T\|_u \leq \frac{c'_9}{c'_8} \|T\|_{u'}.$$

We deduce that T is compact for any $u \in I$. Indeed; , If we consider F , a bounded closed subset in $(\mathcal{H}, < \cdot, \cdot >_u)$ for some $u \in I$, then according to the last inequality, F is bounded and closed for $< \cdot, \cdot >_{u_0}$, and since $T \in \mathcal{C}_{1, u_0}(\mathcal{H})$, then (by definition) T is compact in $(\mathcal{H}, < \cdot, \cdot >_{u_0})$. It follows that $T(F)$ is relatively compact in the previous space. Using the same inequality, we conclude that $T(F)$ is relatively compact in $(\mathcal{H}, < \cdot, \cdot >_u)$.

Let R be a operator of finite rank, with rank less than n . We have

$$\frac{c'_8}{c'_9} \|T - R\|_{u'} \leq \|T - R\|_u \leq \frac{c'_9}{c'_8} \|T - R\|_{u'}.$$

Then,

$$\frac{c'_8}{c'_9} \sigma_n(T)_{u'} \leq \sigma_n(T)_u \leq \frac{c'_9}{c'_8} \sigma_n(T)_{u'}.$$

Therefore

$$\frac{c'_8}{c'_9} \|T\|_{1, u'} \leq \|T\|_{1, u} \leq \frac{c'_9}{c'_8} \|T\|_{1, u'}.$$

□

Corollary 4.11. *Let $(\|\cdot\|_u)_{u \geq 1}$ a sequence of hermitian metrics on \mathcal{H} , which converges uniformly to a hermitian metric denoted $\|\cdot\|_\infty$, when u goes to infinity $\|\cdot\|_\infty$. We suppose that $\forall \varepsilon > 0$, there exists $\eta > 0$ such that*

$$(1 - \varepsilon) \|\xi\|_{u'} \leq \|\xi\|_u \leq (1 + \varepsilon) \|\xi\|_{u'} \quad \forall u, u' > \eta. \quad (27)$$

Then, we have for any $0 < \varepsilon < 1$

$$\frac{1 - \varepsilon}{1 + \varepsilon} \sigma_n(T)_{u'} \leq \sigma_n(T)_u \leq \frac{1 + \varepsilon}{1 - \varepsilon} \sigma_n(T)_{u'} \quad \forall T \in L(\mathcal{H}), \forall n \in \mathbb{N}, \forall u, u' > \eta.$$

In particular,

$$\frac{1 - \varepsilon}{1 + \varepsilon} \|T\|_{1, u'} \leq \|T\|_{1, u} \leq \frac{1 + \varepsilon}{1 - \varepsilon} \|T\|_{1, u'} \quad \forall T \in L(\mathcal{H}), \quad \forall u, u' > \eta.$$

Proof. This is a consequence of the proof of the previous proposition. □

We consider the prehilbertian space $A^{0,0}(X)$, endowed with the metric $\|\cdot\|_{L^2,u}$ associated to the metric $h_{X,u}$. We know that $(h_{X,u})_u$ converges uniformly to a limit $h_{X,\infty}$. Hence $(\|\cdot\|_{L^2,u})$ converges uniformly to $\|\cdot\|_{L^2,\infty}$. (More precisely, it verifies (27)).

Let $T = Be^{-t\Delta_{X,u}}$, where B is a bounded operator and $t > 0$ is fixed. We have, for any $u \gg 1$

$$\frac{1-\varepsilon}{1+\varepsilon} \sigma_n(Be^{-t\Delta_{X,u}})_\infty \leq \sigma_n(Be^{-t\Delta_{X,u}})_u \leq \frac{1+\varepsilon}{1-\varepsilon} \sigma_n(Be^{-t\Delta_{X,u}})_\infty.$$

Therefore,

$$\frac{1-\varepsilon}{1+\varepsilon} \|Be^{-t\Delta_{X,u}}\|_{1,\infty} \leq \|Be^{-t\Delta_{X,u}}\|_{1,u} \leq \frac{1+\varepsilon}{1-\varepsilon} \|Be^{-t\Delta_{X,u}}\|_{1,\infty}. \quad (28)$$

We consider metric L^2_∞ on \mathcal{H} . We have, for any $t > 0$ fixed

$$\begin{aligned} \left| \|Be^{-t\Delta_{X,u}}e^{-t\Delta_{X,\infty}}\|_{1,\infty} - \|Be^{-2t\Delta_{X,\infty}}\|_{1,\infty} \right| &\leq \|Be^{-t\Delta_{X,u}}e^{-t\Delta_{X,\infty}} - Be^{-2t\Delta_{X,\infty}}\|_{1,\infty} \\ &= \|B(e^{-t\Delta_{X,u}} - e^{-t\Delta_{X,\infty}})e^{-t\Delta_{X,\infty}}\|_{1,\infty} \\ &\leq \|e^{-t\Delta_{X,u}} - e^{-t\Delta_{X,\infty}}\|_{L^2,\infty} \|Be^{-t\Delta_{X,\infty}}\|_{1,\infty} \end{aligned} \quad (29)$$

and

$$\begin{aligned} \left| \|Be^{-t\Delta_{X,u}}e^{-t\Delta_{X,\infty}}\|_{1,\infty} - \|Be^{-2t\Delta_{X,u}}\|_{1,\infty} \right| &\leq \|Be^{-t\Delta_{X,u}}e^{-t\Delta_{X,\infty}} - Be^{-2t\Delta_{X,u}}\|_{1,\infty} \\ &\leq \|Be^{-t\Delta_{X,u}}(e^{-t\Delta_{X,u}} - e^{-t\Delta_{X,\infty}})\|_{1,\infty} \\ &\leq \|e^{-t\Delta_{X,u}} - e^{-t\Delta_{X,\infty}}\|_{L^2,\infty} \|Be^{-t\Delta_{X,u}}\|_{1,\infty}. \end{aligned} \quad (30)$$

4.3 The heat kernel of Laplacians

We follow [16, Appendice, D], for the definition of heat kernel associated to a Laplacian. Let Δ be a Laplacian, roughly speaking, one defines heat kernel operator using the theory of operators, and it is denoted by $e^{-t\Delta}$. For any $t > 0$, $e^{-t\Delta}$ is a compact operator on $L^2(X)$ to $L^2(X)$ which is \mathcal{C}^1 in t and which verifies the following properties : for $s \in L^2(X)$

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta \right) e^{-t\Delta} s &= 0, \\ \lim_{t \rightarrow 0} e^{-t\Delta} s &= s \quad \text{dans } L^2(X). \end{aligned}$$

One shows that $e^{-t\Delta}$ is unique.

Theorem 4.12. *Let Δ^u be a smooth family of Laplacians, then for each $t > 0$, the corresponding family of heat kernels $e^{-t\Delta^u}$ defines a smooth family of operators on \mathcal{O} . Furthermore, the derivative of $e^{-t\Delta^u}$ with respect to u is given by Duhamel formula:*

$$\frac{\partial}{\partial u} e^{-t\Delta^u} = - \int_0^t e^{-(t-s)\Delta^u} (\partial_u \Delta^u) e^{-s\Delta^u} ds. \quad (31)$$

Proof. See [16, theorem D.1.6] ou [2, theorem 2.48]. □

Theorem 4.13. *Let V be a Banach space. If A is selfadjoint and positive operator, then $-A$ generates a semi-group $P(t) = e^{-tA}$ consisting of positive and selfadjoint operators with norms ≤ 1 .*

Proof. See [20, Proposition 9.4]. □

5 Appendix 2

In this section, we present some results and technical lemmas needed through this article.

5.1 A technical result

In the following proposition, given a discrete sequence $(h_n)_{n \in \mathbb{N}^*}$, of hermitian metrics on a vector bundle on a Riemannian manifold, we construct a family $(h_u)_{u \in \mathbb{N}^*}$ with continuous parameter which behaves smoothly and preserves some properties of the previous sequence. This construction, will be useful to study the infinitesimal variations of different objects attached to this sequence.

Proposition 5.1. *Let X be a complex manifold. Let E be a holomorphic vector bundle, for example TX . We denote by $\mathcal{Met}(E)$ the space of continuous (integrable, smooth,...) hermitian metrics on E . Let $(h_n)_{n \in \mathbb{N}}$ be a sequence hermitian metrics on E (non necessarily smooth), then there exists a continuous family $(H(u))_{u \geq 1}$ such that:*

1. $H(u)$ is a hermitian metric on E , $\forall u$.
2. For any holomorphic local section s of E , the following application

$$H : [1, \infty[\longrightarrow \mathbb{R}^+ \\ u \longmapsto H(u)(s, s),$$

is smooth.

3. $H(n) = h_n$, $\forall n \in \mathbb{N}$.
4. If we suppose that $(h_n)_{n \in \mathbb{N}}$ converges uniformly to a metric h_∞ , then the family $(\frac{H(u)}{h_\infty})_{u \geq 1}$ of smooth functions on X , converges uniformly to the constant function 1 on X .
5. If E is a line bundle, then

$$\frac{\partial}{\partial u} \log H(u) = O\left(\frac{h_{[u]+1} - h_{[u]}}{h_{[u]}}\right),$$

on X , with $[u]$ is the round up of u .

Proof. Let $(h_p)_{p \geq 2}$ be a discrete sequence of hermitian metrics. For any n , let ρ_n be a smooth, positive and nondecreasing function on \mathbb{R}^+ such that:

$$\rho_n(x) = \begin{cases} 0, & x \leq n \\ 1, & x \geq n+1 \end{cases} \quad (32)$$

We can assume that $\rho_n(x) = \rho_1(x - n)$, $\forall x \in \mathbb{R}^+$.

We set $H_1 : [1, \infty[\longrightarrow \mathcal{Met}$ such that $H_1(u) = h_1$, $\forall u$. If \mathcal{H}_2 is the following function: $\mathcal{H}_2(u) = (1 - \rho_1(u))H_1(u) + \rho_1(u)h_2$ then it is smooth and verify $\mathcal{H}_2(1) = H_1(1) = h_1$ and $\mathcal{H}_2(2) = h_2$. By induction on $k \in \mathbb{N}$, we set $H_k(u) = (1 - \rho_{k-1}(u))H_{k-1}(u) + \rho_{k-1}(u)h_k$, and we have $H_k(i) = h_i$, for $i \leq k-1$ and $H_k(k) = h_k$.

Then, we let $H : \mathbb{R}^+ \longrightarrow \mathcal{Met}(\mathcal{O})$ such that $H(u) = H_n(u)$ if $u \leq n-1$ (remark that $H_{n+1}(u) = H_n(u)$) then H is well defined, smooth and $H(n) = h_n$ for any $n \in \mathbb{N}$.

Suppose that $(h_n)_{n \in \mathbb{N}}$ converges uniformly to h_∞ . One proves, using induction on k , that

$$H(u) = (1 - \rho_{k-1}(u))h_{k-1} + \rho_{k-1}(u)h_k \quad \forall u \in [k-1, k] \quad \forall k \in \mathbb{N}^*. \quad (33)$$

Then, if s a nonzero holomorphic local section of \mathcal{O} on an open subset U , we have:

$$\left| \frac{H(u)(s, s)}{h_\infty(s, s)} - 1 \right| \leq \left| \frac{h_{k-1}(s, s)}{h_\infty(s, s)} - 1 \right| + \left| \frac{h_k(s, s)}{h_\infty(s, s)} - 1 \right| \quad \forall u \in [k-1, k],$$

then we get (4.).

We have, for any $u > 1$:

$$\begin{aligned} \left| \frac{\partial}{\partial u} \log H(u)(s, s) \right| &= |h_u(s, s)^{-1} (\partial_u \rho_{k-1})(u) (h_k(s, s) - h_{k-1}(s, s))| \\ &= |\partial_u \rho_{k-1}(u)| \left| \frac{h_k(s, s) - h_{k-1}(s, s)}{h_u(s, s)} \right| \\ &\leq |\partial_u \rho_{k-1}(u)| \left| \frac{h_k - h_{k-1}}{\min(h_{k-1}, h_k)} \right| \\ &= |\partial_u \rho_{k-1}(u)| \max \left(\frac{|h_k - h_{k-1}|}{h_{k-1}}, \frac{|h_k - h_{k-1}|}{h_k} \right). \end{aligned}$$

Since $|\partial_u \rho_{k-1}(u)|$ is uniformly bounded (take for instance, $\rho_k(x) = \rho_1(x - k)$, for any $k \in \mathbb{N}_{\geq 2}$). Then, there exists a constant $M > 0$ such that:

$$\left| \frac{\partial}{\partial u} \log H(u) \right| \leq M \frac{|h_{[u]+1} - h_{[u]}|}{h_{[u]}},$$

for any $u \geq 1$. □

In this paper, we will denote by h_u , the metric $H(u)$.

5.2 On the first nonzero eigenvalue of the Laplacian

In this section, we recall a result due to Cheeger, which gives a geometric lower bound for the first nonzero eigenvalue of the Laplacian Δ , such that $\overline{\mathcal{O}}_0$ is the trivial line bundle endowed with a constant metric. This lower bound can be expressed in terms of the geometry of the manifold, see the theorem below.

Let $(h_{X,p})_{p \in \mathbb{N}}$ be a bounded sequence of smooth hermitian metrics on X , if we denote by $\lambda_{p,1}$, for any $p \in \mathbb{N}$, the first nonzero eigenvalue of the Laplacian attached to $((TX, h_{X,p}), \overline{\mathcal{O}}_0)$, then will show there exists a constant $\kappa > 0$ such that

$$\lambda_{p,1} \geq \kappa, \quad \forall p \in \mathbb{N}.$$

Definition 5.2 (Cheeger isoperimetric constant). Let (M, g) be a compact riemannian manifold of dimension n without boundary. We set:

$$h_g(M) := \inf \frac{A(S)}{\min(V(M_1), V(M_2))},$$

where $A(\cdot)$ denote the $(n-1)$ dimensional volume, $V(\cdot)$ is the volume and the inf is taken over the set of compact submanifolds with corners S of dimension $n-1$, M_1 and M_2 are the two open submanifolds with boundary such that $M = M_1 \cup M_2$ and $\partial M_j = S$, for $j = 1, 2$.

Theorem 5.3. Let λ_1 be the first nonzero eigenvalue of the Laplacian associated to (M, g) , then

$$\lambda_1 \geq \frac{1}{4} h_g^2(M).$$

Proof. See [5]. □

Remark 5.4. 1. $h_{tg}(M) = t^{-\frac{1}{2}}h_g(M)$, for any $t > 0$. (follows from the definition of $h_g(M)$).

2. $h_g(M)$ is a positive real, see [5, p.198] for $n = 2$.

Proposition 5.5. *Let us consider a bounded sequence $(g_p)_{p \in \mathbb{N}_{\geq 2}}$ of smooth riemannian metrics on X , namely, we suppose there exist c_1 and c_2 two nonzero constants such that:*

$$c_1 g_p \leq g_q \leq c_2 g_p \quad \forall p, q \in \mathbb{N}. \quad (34)$$

Then

$$\frac{c_1}{c_2} h_{g_p}(M) \leq h_{g_q}(M) \leq \frac{c_2}{c_1} h_{g_p}(M) \quad \forall p, q \in \mathbb{N}. \quad (35)$$

Proof. It suffices to note that (34) is stable by restriction to submanifolds of M . □

Proposition 5.6. *Let $(g_p)_{p \in \mathbb{N}}$ be as before, then there exists κ a positive constant such that:*

$$\lambda_{p,1} \geq \kappa \quad \forall p \in \mathbb{N},$$

where $\lambda_{p,1}$ is the first nonzero eigenvalue of the Laplacian associated to g_p .

Proof. It follows from (5.3) and (5.5). □

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